Singularities in Foliations (Distributions) Defined by Normal Curvature Properties

J. Sotomayor

Instituto de Matemática e Estatística
Universidade de São Paulo

`sotpj@yahoo.com.br`

Work in collaboration with R. A. Garcia (UFG) and D. Lopes da Silva (UFS).

`Bogotá, Taller` 2^{nd} `and` 3^{rd} `Lectures.`

October, 2012

DEDICATED TO THE INTERACTION OF
GEOMETRY AND DIFFERENTIAL EQUATIONS.

RECALLING A VERY HAPPY OCCASION:

DEDICATED TO THE INTERACTION OF
GEOMETRY AND DIFFERENTIAL EQUATIONS.

RECALLING A VERY HAPPY OCCASION:

The Fiftieth Aniversary of

Peixoto's Structurally Stability Theorem
for Generic Vector Fields on 2 - Manifolds.

I- Generalities and Plan for Today's Lecture.

I will focus on results that evolved from my collaboration with Carlos Gutiérrez (1944 - 2008), which began to be published in 1982, Asterisque, dealing with the Structurally Stable Principal Curvature Configurations on Surfaces in $\mathbb{R}^3$, under small perturbations of their immersions.

This is our link with Peixoto's Theorem for Structurally Stable Vector Fields on Surfaces, historical landmark in the Geometric Theory of Differential Equations and Dynamical Systems. Topology 1962.

The Principal Configuration on an Immersed Surface is the counterpart of the Phase Portrait of an ODE, thought for the case of the two families of principal curvature lines – maximal and minimal – on a surface in $\mathbb{R}^3$.

This presentation connects our subject with the works on Principal Curvature Configurations of remarkable mathematicians of a more distant past,
such as:

Monge, after Euler

Dupin,   Theory I.

and

Darboux,   Theory II, after Poincaré.

Here would come Peixoto's Contribution, after Andronov - Pontrjagin $\cdots$

Pictorial Outline: Euler, Normal Curvature and Principal Directions.

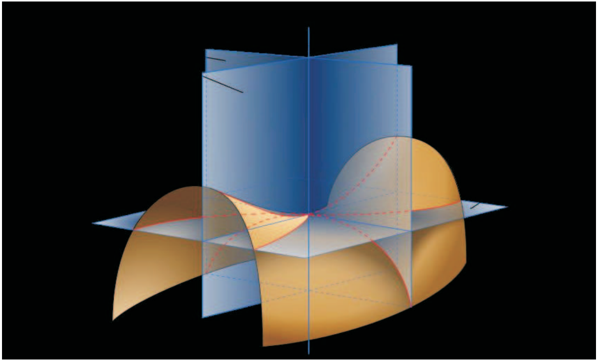


Figure: Principal Directions and Curvatures.

Pictorial Outline: Monge, 1796. First Prin. Config.

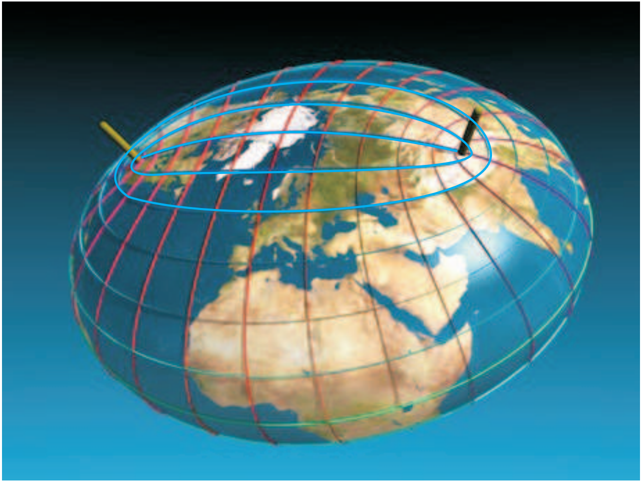


Figure: Monge's Ellipsoid: First Principal Config. Illustration from E. Ghys site.

Pictorial Outline: Dupin. First Qualitative Theory for Prin. Configs, 1818.

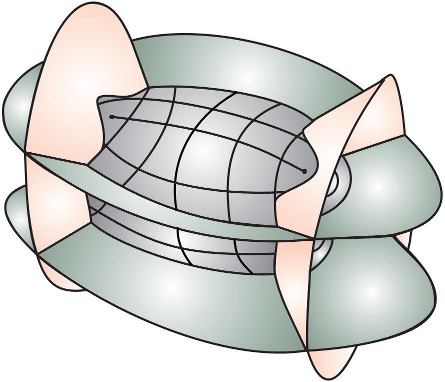


Figure: Dupin's Theorem. Integrable Principal Configurations.

Pictorial Outline: G. Darboux. A Generic Theory for Princ. Configs. near Umbilic Points, 1896.

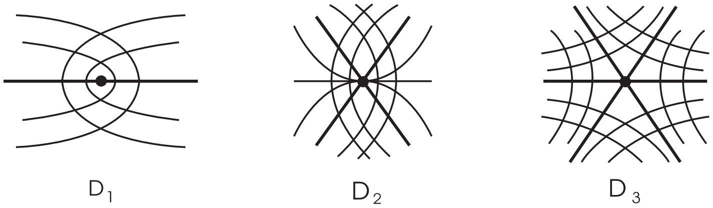
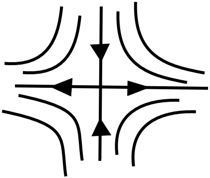
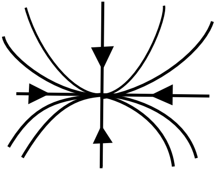


Figure: Darbouxian Umbilic Points for C^ω, after Poincaré's ODEs, 1881, and Soto-Gutierrez for C^4, 1982. Definition in a Monge chart below.

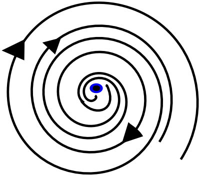
After Poincaré. Singularities and MORE...



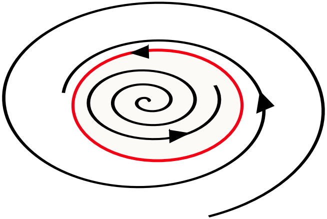
saddle



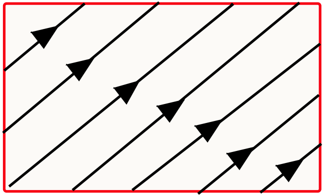
attracting node



focus



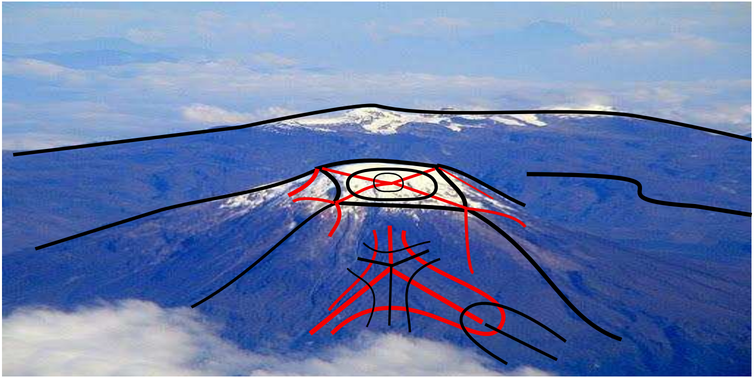
limit cycle



linear flow on the Torus.

Pão de Açúcar

Monte Tolima



II. Analytic Outline of Umbilic Singularities.

A local Monge chart near an umbilic point

is given by $\alpha(u, v) = (u, v, h(u, v))$, where

$$h(u,v) = \frac{k}{2}(u^2+v^2) + \frac{a}{6}u^3 + \frac{b}{2}uv^2 + \frac{c}{6}v^3 + \frac{A}{24}u^4 + \frac{B}{6}u^3v + \frac{C}{4}u^2v^2 + \frac{D}{6}uv^3 + \frac{E}{24}v^4 + \frac{a_{50}}{120}u^5 + \frac{a_{41}}{24}u^4v + \frac{a_{32}}{12}u^3v^2 + \frac{a_{23}}{12}u^2v^3 + \frac{a_{14}}{24}uv^4 + \frac{a_{05}}{120}v^5 + O(6). \tag{1}$$

Question: How to relate these coefficients with the previous pictures?

Classification of umbilic points on surfaces. Analytic and Pictorial.

Darbouxian umbilic points. Classification depends on the 3−jet.

T) Transversality Condition: $b(b-a) \neq 0$; and

D) Discriminant Conditions:

D_1) $\frac{a}{b} > \left(\frac{c}{2b}\right)^2 + 2$; or

D_2) $1 < \frac{a}{b} < \left(\frac{c}{2b}\right)^2 + 2$, or $a \neq 2b$; or

D_3) $\frac{a}{b} < 1$.

Classification of umbilic points on surfaces. Analytic and Pictorial.

Darbouxian umbilic points. Classification depends on the 3−jet.

T) Transversality Condition: $b(b-a) \neq 0$; and

D) Discriminant Conditions:

D_1) $\frac{a}{b} > \left(\frac{c}{2b}\right)^2 + 2$; or

D_2) $1 < \frac{a}{b} < \left(\frac{c}{2b}\right)^2 + 2$, or $a \neq 2b$; or

D_3) $\frac{a}{b} < 1$.

Semi-Darbouxian umbilic points: Cod. 1 Bifurcations.

D_{12}) $cb(b-a) \neq 0$ and,

- or $\frac{a}{b} = \left(\frac{c}{2b}\right)^2 + 2$
- or $\frac{a}{b} = 2$.

D_{23}) $b = a \neq 0$ and $\chi = cB - (C - A + 2k^3)b \neq 0$.

- Notice that D_{23}) depends on the 4−jet.

Differential equations for principal lines. Gluing Element

- $n = 2$ (In classical notation, Singularities happen at Umbilic Points):

$$(Fg - Gf)dv^2 + (Eg - eG)dudv + (Ef - eF)du^2 = 0,$$

$$L = Fg - Gf, \;\; M = Eg - eG, \;\; N = Ef - eF.$$

Umbilic Points, where $k_1 = k_2$, occur at $L = M = N = 0$.

Differential equations for principal lines. Gluing Element

- $n = 2$ (In classical notation, Singularities happen at Umbilic Points):

$$(Fg - Gf)dv^2 + (Eg - eG)dudv + (Ef - eF)du^2 = 0,$$

$$L = Fg - Gf, \quad M = Eg - eG, \quad N = Ef - eF.$$

Umbilic Points, where $k_1 = k_2$, occur at $L = M = N = 0$.

- $n = 3$: For later consideration, essential for today's lecture. More complicated system of IMPLICIT DIFFERENTIAL EQUATIONS.

$$\left\{ \begin{array}{l} (\lambda_{11} - k_i g_{11})du_1 + (\lambda_{12} - k_i g_{12})du_2 + (\lambda_{13} - k_i g_{13})du_3 = 0 \\ (\lambda_{12} - k_i g_{12})du_1 + (\lambda_{22} - k_i g_{22})du_2 + (\lambda_{23} - k_i g_{23})du_3 = 0 \\ (\lambda_{13} - k_i g_{13})du_1 + (\lambda_{23} - k_i g_{23})du_2 + (\lambda_{33} - k_i g_{33})du_3 = 0 \end{array} \right. , \tag{2}$$

where k_i $(i = 1, 2, 3)$ are the principal curvatures, defined by the 3 roots of the cubic equation $\det(\Lambda - kG) = 0$.

Umbilic Points on Surfaces of $\mathbb{R}^3$

Umbilic Points on Surfaces of $\mathbb{R}^3$

Darbouxian Umbilic Points (Generic, structurally stable)

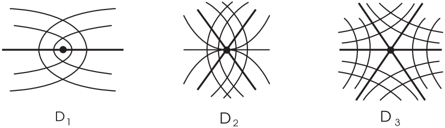
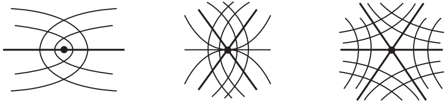


Figure: Darboux - Poincaré, 1881, C^{ω}; Gutierrez-Sotomayor, 1982, C^4.

Umbilic Points on Surfaces of $\mathbb{R}^3$

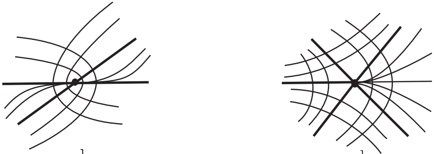
Darbouxian Umbilic Points (Generic, structurally stable)



D_1 D_2 D_3

Figure: Darboux - Poincaré, 1881, C^ω; Gutierrez-Sotomayor, 1982, C^4.

Umbilics of Codimension 1: Bifurcate Generically, one parameter.



D^1_2 $D^1_{2,3}$

Garcia, Gutiérrez, Sotomayor; A. Gullstrand, Ophtalmologist Nobel Laureate, 1904.

Lie-Cartan vector field

The local configuration near umbilics can be explained in terms of the phase portraits of the singularities, hyperbolic saddles and nodes in the present Darbouxian case, of the Lie-Cartan vector field suspension, $X_{\mathcal{F}}$, of the implicit differential equation $\mathcal{F}(u, v, p) = 0$, where,

$$\mathcal{F}(u, v, p) = Lp^2 + Mp + N = 0, \; p = \frac{dv}{du} \qquad (3)$$
$$X_{\mathcal{F}} = (\mathcal{F}_p, p\mathcal{F}_p, -(\mathcal{F}_u + p\mathcal{F}_v))$$

Similar for affine chart $q = 1/p$ to capture behavior at $p = \infty$.

The integral curves to $X_{\mathcal{F}}$ are tangent to $\mathcal{F}(u, v, p) = 0$ and project onto principal curvature lines.

Darbouxian Umbilic points RESOLVED in terms of resolution of hyperbolic singularities of vector fields

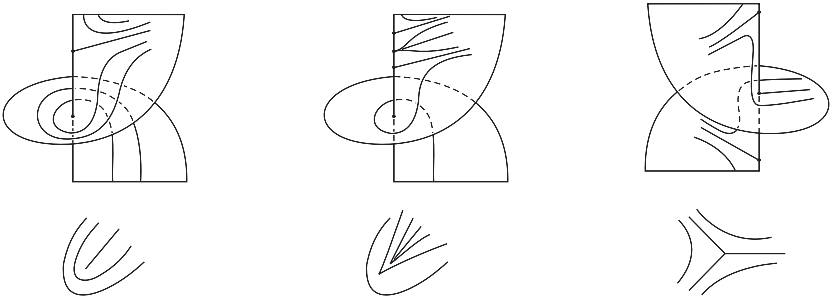
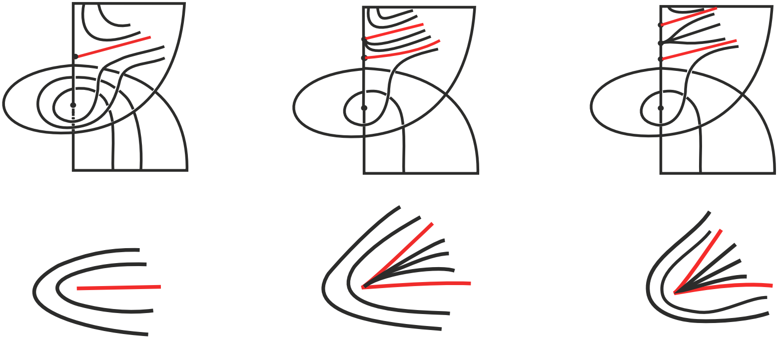
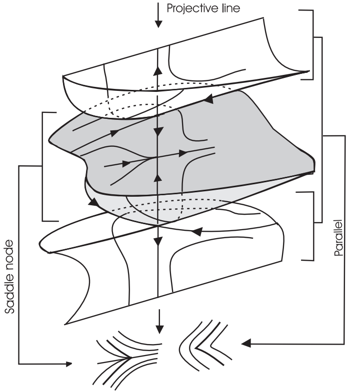


Figure: Artistic Illustration of surfaces over configurations D_1 and D_2. Case D_3 more exact drawing. Attention: gluing at ∞.

Umbilic point D^1_{12} bifurcation (emergence of nodal sector), RESOLVED in terms of hyperbolic and saddle-node singularities of vector fields.



Umbilic point D_{23} RESOLVED in terms of semi-hyperbolic singularities of vector fields defined on a VARIETY with two Morse Critical Points.



Bifurcation: Elimination - Splitting of umbilic points D_2 and D_3 at D_{23}

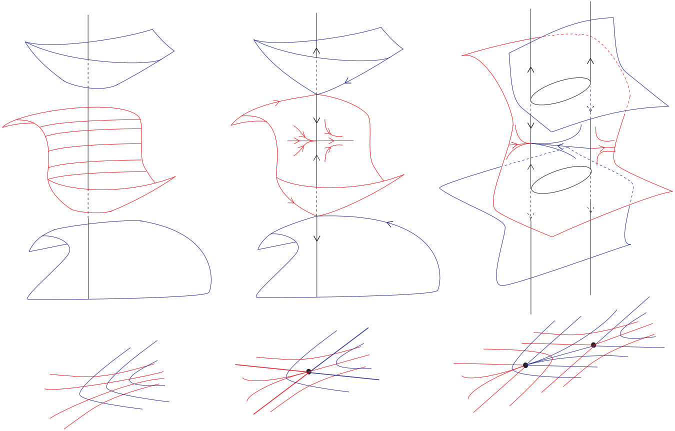
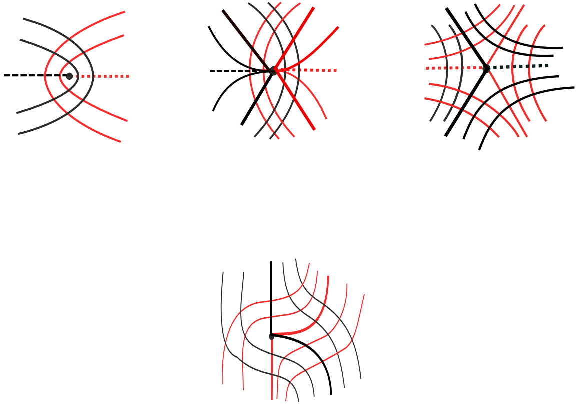


Figure: Bifurcation: Elimination - Splitting of a pair of D_2 of D_3 from a D_{23}.

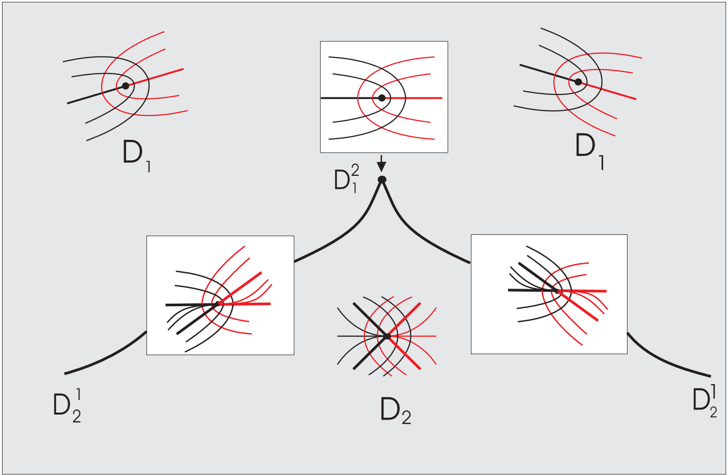
Umbilics of Codimension 2: classification depends on the 5-jet of h

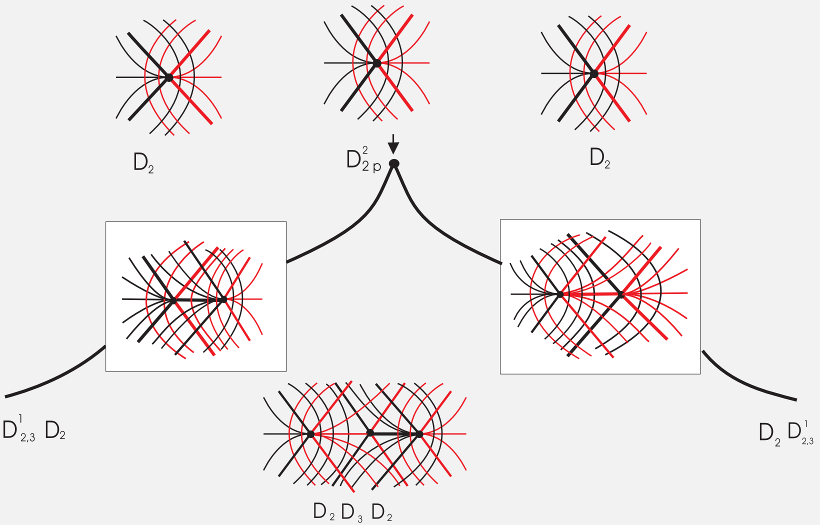
. PICTORIAL OUTLINE

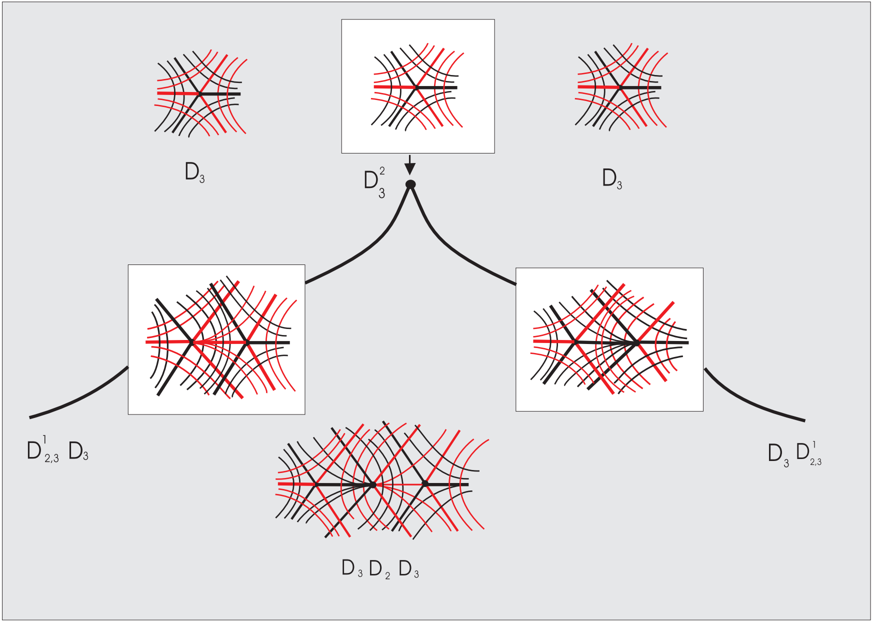


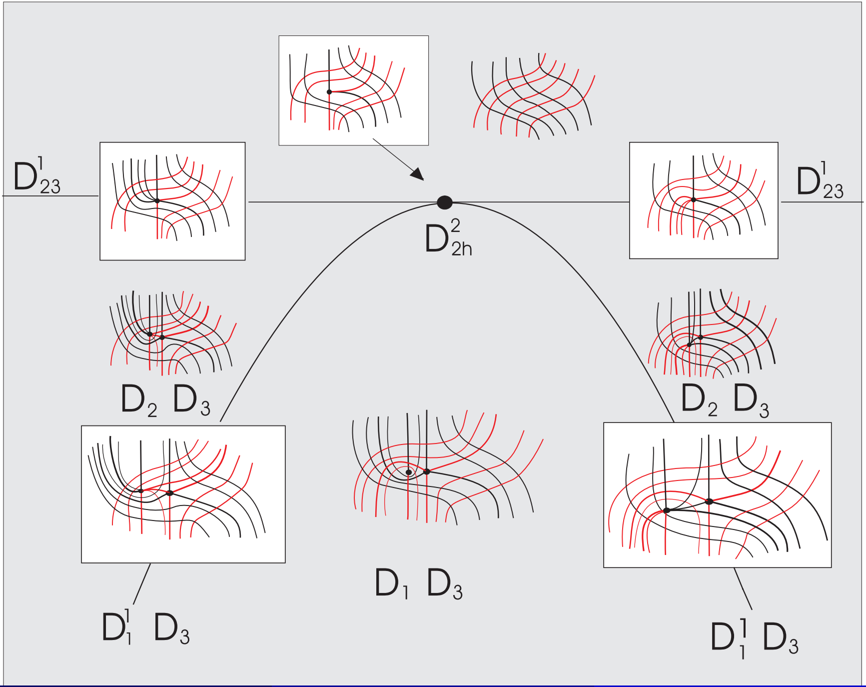
Pictures. Analysis after Lie - Cartan in [Garcia, Sotomayor], discussion below.

Bifurcation diagrams of umbilics of codimension two, [Ga,So]




D_2
$D_{2\,p}^{2}$
D_2
$D^{1}_{2,3}\ D_2$
$D_2\ D_3\ D_2$
$D_2\ D^{1}_{2,3}$


D_3
D_3^2
D_3
$D^1_{2,3}\ D_3$
$D_3\ D^1_{2,3}$
$D_3\ D_2\ D_3$



D^1_{23}
D^1_{23}
D^2_{2h}
$D_2\ D_3$
$D_2\ D_3$
$D_1\ D_3$
$D^1_1\ D_3$
$D^1_1\ D_3$

CONSIDER: Surfaces and Hypersurfaces in $\mathbb{R}^4$.

Follow today the second path: links for references in my webpage:
http://www.ime.usp.br/˜sotp/

Survey, 2008, São Paulo Journ. Math. Also in the arXiv.

Book, 2009, Brazilian Math. Colloquium.

Principal curvature lines on hypersurfaces of $\mathbb{R}^{n+1}$

Principal curvature lines on hypersurfaces of $\mathbb{R}^{n+1}$

For any oriented n-dimensional manifold $\mathbb{M}^n$ immersed in $\mathbb{R}^{n+1}$ are associated n, mutually orthogonal, one dimensional foliations with singularities, called here *principal foliations*, whose leaves are the *principal curvature lines*.

Principal curvature lines on hypersurfaces of $\mathbb{R}^{n+1}$

For any oriented n-dimensional manifold $\mathbb{M}^n$ immersed in $\mathbb{R}^{n+1}$ are associated n, mutually orthogonal, one dimensional foliations with singularities, called here *principal foliations*, whose leaves are the *principal curvature lines*.

These lines are the integral curves of the *principal directions fields* which are defined by the directions where the normal curvature

$$k_n(p, v) = \frac{II_\alpha}{I_\alpha} = \frac{\sum \lambda_{ij} du_i du_j}{\sum g_{ij} du_i du_j}, \qquad \alpha : \mathbb{M}^n \to \mathbb{R}^{n+1}$$

of the immersed hypersurface is critical.

Principal curvature lines on hypersurfaces of $\mathbb{R}^{n+1}$

For any oriented n-dimensional manifold $\mathbb{M}^n$ immersed in $\mathbb{R}^{n+1}$ are associated n, mutually orthogonal, one dimensional foliations with singularities, called here *principal foliations*, whose leaves are the *principal curvature lines*.

These lines are the integral curves of the *principal directions fields* which are defined by the directions where the normal curvature

$$k_n(p, v) = \frac{II_\alpha}{I_\alpha} = \frac{\sum \lambda_{ij} du_i du_j}{\sum g_{ij} du_i du_j}, \qquad \alpha : \mathbb{M}^n \to \mathbb{R}^{n+1}$$

of the immersed hypersurface is critical. The principal curvatures will be denoted by $k_1 \leq k_2 \leq \ldots \leq k_n$ and the principal directions will be denoted by $\{e_1, e_2, \ldots, e_n\}$. These are non oriented lines.

Differential equation of principal lines.

Differential equation of principal lines.

- $n = 2$ (In classical notation for FUNDAMENTAL FORMS.):

$$(Fg - Gf)dv^2 + (Eg - eG)dudv + (Ef - eF)du^2 = 0,$$

$$L = Fg - Gf, \quad M = Eg - eG, \quad N = Ef - eF.$$

Umbilic Points, where $k_1 = k_2$, occur at $L = M = N = 0$.

- $n = 3$: More complicated system of IMPLICIT DIFFERENTIAL EQUATIONS.

$$\left\{ \begin{array}{l} (\lambda_{11} - k_i g_{11})du_1 + (\lambda_{12} - k_i g_{12})du_2 + (\lambda_{13} - k_i g_{13})du_3 = 0 \\ (\lambda_{12} - k_i g_{12})du_1 + (\lambda_{22} - k_i g_{22})du_2 + (\lambda_{23} - k_i g_{23})du_3 = 0 \\ (\lambda_{13} - k_i g_{13})du_1 + (\lambda_{23} - k_i g_{23})du_2 + (\lambda_{33} - k_i g_{33})du_3 = 0 \end{array} \right. , \tag{4}$$

where k_i $(i = 1, 2, 3)$ are the principal curvatures, defined by the 3 roots of the cubic equation $\det(\Lambda - kG) = 0$.

III. Case $n = 3$: Principal curvature lines on hypersurfaces of $\mathbb{R}^4$.

- Singularities:

III. Case $n = 3$: Principal curvature lines on hypersurfaces of $\mathbb{R}^4$.

- Singularities: $\left\{ \begin{array}{l} \textbf{(Totally) Umbilic Points} : k_1(p) = k_2(p) = k_3(p) \\ \end{array} \right.$

III. Case $n = 3$: Principal curvature lines on hypersurfaces of $\mathbb{R}^4$.

- Singularities: $\begin{cases} \textbf{(Totally) Umbilic Points} : k_1(p) = k_2(p) = k_3(p) \\ \textbf{Partially umbilic points} : k_1(p) = k_2(p) \neq k_3(p) \end{cases}$

III. Case $n = 3$: Principal curvature lines on hypersurfaces of $\mathbb{R}^4$.

- Singularities: $\begin{cases} \textbf{(Totally) Umbilic Points} : k_1(p) = k_2(p) = k_3(p) \\ \textbf{Partially umbilic points} : k_1(p) = k_2(p) \neq k_3(p) \end{cases}$

Similar for $k_1(p) \neq k_2(p) = k_3(p)$.

III. Case $n = 3$: Principal curvature lines on hypersurfaces of $\mathbb{R}^4$.

- Singularities: $\begin{cases} \textbf{(Totally) Umbilic Points}: k_1(p) = k_2(p) = k_3(p) \\ \textbf{Partially umbilic points}: k_1(p) = k_2(p) \neq k_3(p) \end{cases}$

Similar for $k_1(p) \neq k_2(p) = k_3(p)$.

The goal is to describe the local behavior of principal foliations near partially umbilic points.

III. Case $n = 3$: Principal curvature lines on hypersurfaces of $\mathbb{R}^4$.

- Singularities: $\begin{cases} \textbf{(Totally) Umbilic Points} : k_1(p) = k_2(p) = k_3(p) \\ \textbf{Partially umbilic points} : k_1(p) = k_2(p) \neq k_3(p) \end{cases}$

Similar for $k_1(p) \neq k_2(p) = k_3(p)$.

The goal is to describe the local behavior of principal foliations near partially umbilic points.

In fact: Generically (in the space of immersions) the set of (totally) umbilic points is empty (Cod 5) and the set of partially umbilic points (Cod. 2) is the union of regular curves on $\mathbb{M}$.

III. Case $n = 3$: Principal curvature lines on hypersurfaces of $\mathbb{R}^4$.

- Singularities: $\begin{cases} \textbf{(Totally) Umbilic Points} : k_1(p) = k_2(p) = k_3(p) \\ \textbf{Partially umbilic points} : k_1(p) = k_2(p) \neq k_3(p) \end{cases}$

Similar for $k_1(p) \neq k_2(p) = k_3(p)$.

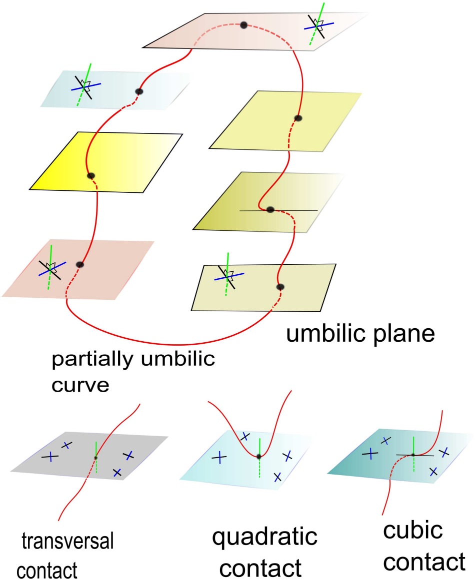
The goal is to describe the local behavior of principal foliations near partially umbilic points.

In fact: Generically (in the space of immersions) the set of (totally) umbilic points is empty (Cod 5) and the set of partially umbilic points (Cod. 2) is the union of regular curves on $\mathbb{M}$.

REGULAR POINTS: $k_1(p) < k_2(p) < k_3(p)$.

Partially umbilic curve and contact with the umbilic plane

Partially umbilic curve and contact with the umbilic plane



Plane field defined by a regular principal direction e_3 near a partially umbilic point

Consider the plane passing through $q \in M$ having the principal direction $e_3(q)$ as the normal vector.

$$\Pi(q) = \{(du_1, du_2, du_3); \left\langle (du_1, du_2, du_3), G \cdot (e_3(q))^T \right\rangle = 0\}, \quad (5)$$

where $G = [g_{ij}]_{3\times 3}$ is the first fundamental form.

Therefore the plane field Π is defined by the differential one form

Plane field defined by a regular principal direction e_3 near a partially umbilic point

Consider the plane passing through $q \in M$ having the principal direction $e_3(q)$ as the normal vector.

$$\Pi(q) = \{(du_1, du_2, du_3); \left\langle (du_1, du_2, du_3), G \cdot (e_3(q))^T \right\rangle = 0\}, \qquad (5)$$

where $G = [g_{ij}]_{3\times 3}$ is the first fundamental form.

Therefore the plane field Π is defined by the differential one form

$$du_3 = \mathcal{U}(u_1, u_2, u_3)du_1 + \mathcal{V}(u_1, u_2, u_3)du_2. \qquad (6)$$

Plane field defined by a regular principal direction e_3 near a partially umbilic point

Consider the plane passing through $q \in M$ having the principal direction $e_3(q)$ as the normal vector.

$$\Pi(q) = \{(du_1, du_2, du_3); \left\langle (du_1, du_2, du_3), G \cdot (e_3(q))^T \right\rangle = 0\}, \qquad (5)$$

where $G = [g_{ij}]_{3\times 3}$ is the first fundamental form.

Therefore the plane field Π is defined by the differential one form

$$du_3 = \mathcal{U}(u_1, u_2, u_3)du_1 + \mathcal{V}(u_1, u_2, u_3)du_2. \qquad (6)$$

The plane field Π is in general not integrable (Frobenious) and so the situation is strictly three-dimensional.

The principal directions $e_1(q)$ and $e_2(q)$ associated to $k_1(q)$ and $k_2(q)$ belong to the plane $\Pi(q)$.
Let

$$\begin{aligned} I_r(du_1, du_2) &= I_\alpha\Big|_{du_3 = \mathcal{U}(u_1,u_2,u_3)du_1 + \mathcal{V}(u_1,u_2,u_3)du_2} \\ &= E_r du_1^2 + 2F_r du_1 du_2 + G_r du_2^2, \\ II_r(du_1, du_2) &= II_\alpha\Big|_{du_3 = \mathcal{U}(u_1,u_2,u_3)du_1 + \mathcal{V}(u_1,u_2,u_3)du_2} \\ &= e_r du_1^2 + 2f_r du_1 du_2 + g_r du_2^2, \end{aligned}$$

The principal directions $e_1(q)$ and $e_2(q)$ associated to $k_1(q)$ and $k_2(q)$ belong to the plane $\Pi(q)$.
Let

$$\begin{aligned} I_r(du_1, du_2) &= I_\alpha\Big|_{du_3=\mathcal{U}(u_1,u_2,u_3)du_1+\mathcal{V}(u_1,u_2,u_3)du_2} \\ &= E_r du_1^2 + 2F_r du_1 du_2 + G_r du_2^2, \\ II_r(du_1, du_2) &= II_\alpha\Big|_{du_3=\mathcal{U}(u_1,u_2,u_3)du_1+\mathcal{V}(u_1,u_2,u_3)du_2} \\ &= e_r du_1^2 + 2f_r du_1 du_2 + g_r du_2^2, \end{aligned}$$

We have that $k_n^r(q,\cdot) = \dfrac{II_r}{I_r}(q,\cdot)$ where $I_r(q)$ and $II_r(q)$ are the first and second fundamental forms of α restricted to the plane $\Pi(q)$.

The principal directions $e_1(q)$ and $e_2(q)$ associated to $k_1(q)$ and $k_2(q)$ belong to the plane $\Pi(q)$.
Let

$$\begin{aligned} I_r(du_1, du_2) &= I_\alpha\Big|_{du_3=\mathcal{U}(u_1,u_2,u_3)du_1+\mathcal{V}(u_1,u_2,u_3)du_2} \\ &= E_r du_1^2 + 2F_r du_1 du_2 + G_r du_2^2, \\ II_r(du_1, du_2) &= II_\alpha\Big|_{du_3=\mathcal{U}(u_1,u_2,u_3)du_1+\mathcal{V}(u_1,u_2,u_3)du_2} \\ &= e_r du_1^2 + 2f_r du_1 du_2 + g_r du_2^2, \end{aligned}$$

We have that $k_n^r(q,\cdot) = \dfrac{II_r}{I_r}(q,\cdot)$ where $I_r(q)$ and $II_r(q)$ are the first and second fundamental forms of α restricted to the plane $\Pi(q)$.
Let $P = \dfrac{du_2}{du_1}$. Therefore the directions $e_1(q)$ and $e_2(q)$ are defined by

$$L_r(u_1, u_2, u_3)P^2 + M_r(u_1, u_2, u_3)P + N_r(u_1, u_2, u_3) = 0, \tag{7}$$

$$L_r = F_r g_r - f_r G_r, \quad M_r = E_r g_r - e_r G_r, \quad N_r = E_r f_r - e_r F_r$$

Let $\mathcal{L}(u_1, u_2, u_3, P) = L_r(u_1, u_2, u_3)P^2 + M_r(u_1, u_2, u_3)P + N_r(u_1, u_2, u_3)$.
The equation

$$\mathcal{L}(u_1, u_2, u_3; P) = 0 \tag{8}$$

defines a hypersurface (variety) in the projective tangent bundle $\mathbb{PM}^3$, called de Lie-Cartan hypersurface, and under generic conditions $b(b-a) \neq 0$ is regular.
The Lie-Cartan vector field suspension is given by:

Let $\mathcal{L}(u_1, u_2, u_3, P) = L_r(u_1, u_2, u_3)P^2 + M_r(u_1, u_2, u_3)P + N_r(u_1, u_2, u_3)$. The equation

$$\mathcal{L}(u_1, u_2, u_3; P) = 0 \tag{8}$$

defines a hypersurface (variety) in the projective tangent bundle $\mathbb{PM}^3$, called de Lie-Cartan hypersurface, and under generic conditions $b(b-a) \neq 0$ is regular.
The Lie-Cartan vector field suspension is given by:

$$X := \begin{cases} \dot{u}_1 &= \mathcal{L}_P \\ \dot{u}_2 &= P\mathcal{L}_P \\ \dot{u}_3 &= (\mathcal{U} + \mathcal{V}P)\,\mathcal{L}_P \\ \dot{P} &= -\left(\mathcal{L}_{u_1} + P\mathcal{L}_{u_2} + \mathcal{L}_{u_3}(\mathcal{U} + \mathcal{V}P)\right) \end{cases}, \tag{9}$$

A Monge Chart near a partially umbilic point

A Monge Chart near a partially umbilic point

$\alpha(u_1, u_2, u_3) = (u_1, u_2, u_3, h(u_1, u_2, u_3))$ where

A Monge Chart near a partially umbilic point

$\alpha(u_1, u_2, u_3) = (u_1, u_2, u_3, h(u_1, u_2, u_3))$ where

$$
\begin{aligned}
h(u_1, u_2, u_3) = {} & \tfrac{k}{2}(u_1^2 + u_2^2) + \tfrac{k_3}{2} u_3^2 + \tfrac{a}{6} u_1^3 + \tfrac{b}{2} u_1 u_2^2 + 0 u_1^2 u_2 + \tfrac{c}{6} u_2^3 \\
& + u_3 [\tfrac{q_{201}}{2} u_1^2 + q_{111} u_1 u_2 + \tfrac{q_{021}}{2} u_2^2] + \tfrac{q_{102}}{2} u_1 u_3^2 + \tfrac{q_{003}}{6} u_3^3 \\
& + \tfrac{q_{012}}{2} u_2 u_3^2 + \tfrac{A}{24} u_1^4 + \tfrac{B}{6} u_1^3 u_2 + \tfrac{C}{4} u_1^2 u_2^2 + \tfrac{D}{6} u_1 u_2^3 \\
& + \tfrac{E}{24} u_2^4 + \tfrac{Q_{004}}{24} u_3^4 + \tfrac{Q_{013}}{6} u_3^3 u_2 + \tfrac{Q_{103}}{6} u_3^3 u_1 \\
& + \tfrac{Q_{022}}{4} u_2^2 u_3^2 + \tfrac{Q_{202}}{4} u_1^2 u_3^2 + \tfrac{Q_{112}}{2} u_3^2 u_1 u_2 + \tfrac{Q_{031}}{6} u_3 u_2^3 \\
& + \tfrac{Q_{301}}{6} u_1^3 u_3 + \tfrac{Q_{121}}{2} u_3 u_2^2 u_1 + \tfrac{Q_{211}}{2} u_3 u_2 u_1^2 + O(5)
\end{aligned}
\tag{10}
$$

Parametrization of a partially umbilic line.

It is defined by $L_r = 0, \; M_r = 0.$

Parametrization of a partially umbilic line.

It is defined by $L_r = 0, \; M_r = 0$. Under Darbouxian transversality condition, the partially umbilic set can be parameterized by

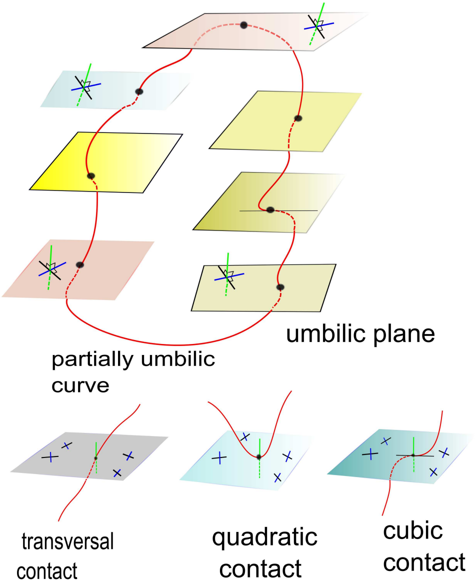
$$u_1 = c_1(u_3), u_2 = c_2(u_3) \text{ defined by} L_r(u_1, u_2, u_3) = M_r(u_1, u_2, u_3) = 0 \tag{11}$$

where,

$$c'(0) = \left[\frac{(-cq_{111} + q_{021}b - q_{201}b)}{b(-b+a)}, -\frac{q_{111}}{b}, 1 \right].$$

Partially umbilic curve: contact with the plane $\Pi(q)$

Partially umbilic curve: contact with the plane $\Pi(q)$



Generic Partially Umbilic Points.

Generic Partially Umbilic Points.

Darbouxian partially umbilic point.

T) $b(b-a) \neq 0$; and

D_1) $\frac{a}{b} > \left(\frac{c}{2b}\right)^2 + 2$; or

D_2) $1 < \frac{a}{b} < \left(\frac{c}{2b}\right)^2 + 2$, $a \neq 2b$; or

D_3) $\frac{a}{b} < 1$.

Generic Partially Umbilic Points.

Darbouxian partially umbilic point.

$T)$ $b(b-a) \neq 0$; and

$D_1)$ $\frac{a}{b} > \left(\frac{c}{2b}\right)^2 + 2$; or

$D_2)$ $1 < \frac{a}{b} < \left(\frac{c}{2b}\right)^2 + 2$, $a \neq 2b$; or

$D_3)$ $\frac{a}{b} < 1$.

Semi-Darbouxian partially umbilic point.

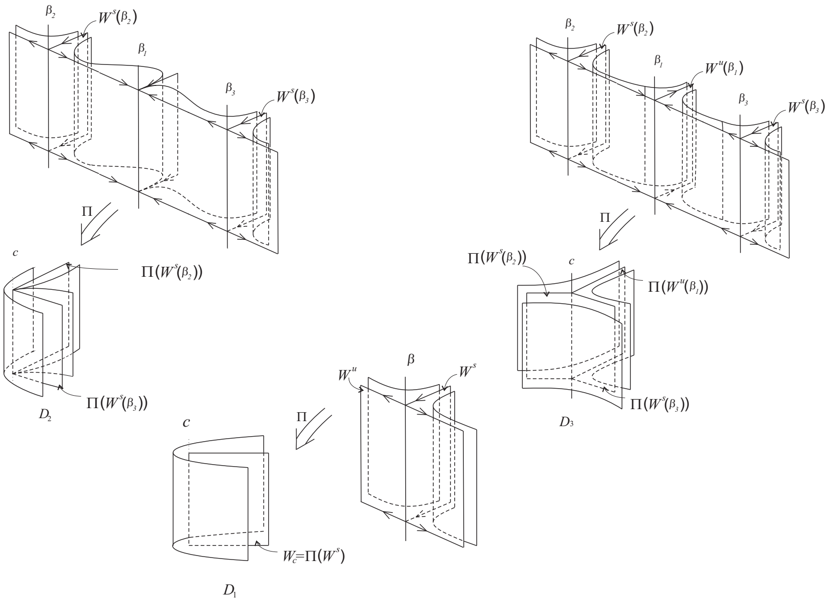
$D_{12})$ $cb(b-a) \neq 0$ and,

- or $\frac{a}{b} = \left(\frac{c}{2b}\right)^2 + 2$ e $\chi_1 \neq 0$,
- or $\frac{a}{b} = 2$, and $\chi_2 \neq 0$.

$D_{23})$ $b = a \neq 0$ and

$$\chi = cB - (C - A + 2k^3)b + \frac{-2q_{111}^2 b + 2q_{201}^2 b + 2q_{201} q_{111} c}{b(k-k_3)} \neq 0.$$

Resolution of Partially umbilic points.



Resolution of Partially umbilic points.

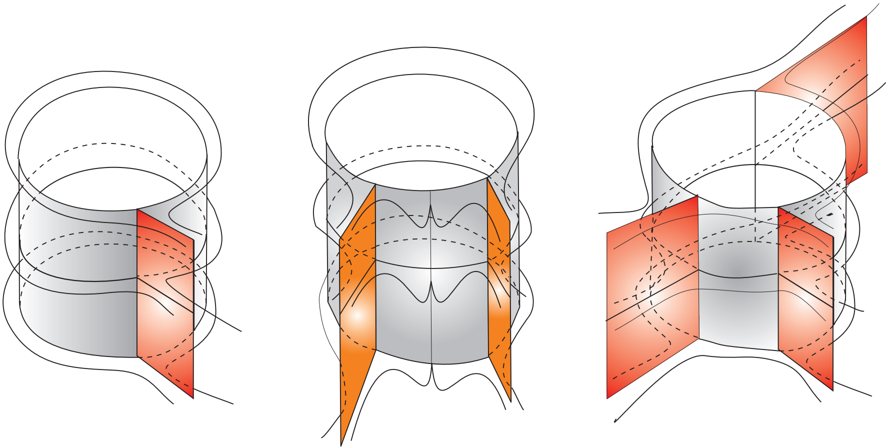
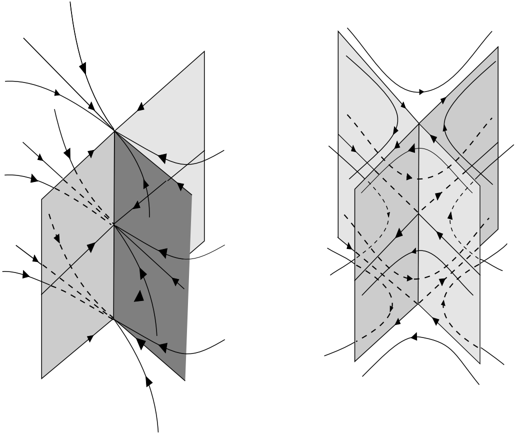
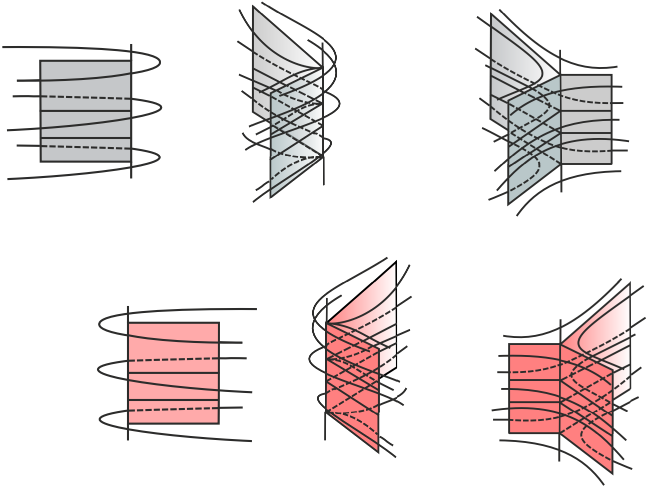


Figure: D_1, D_2, D_3

Resolution of Partially umbilic points in terms of normally hyperbolic manifolds.

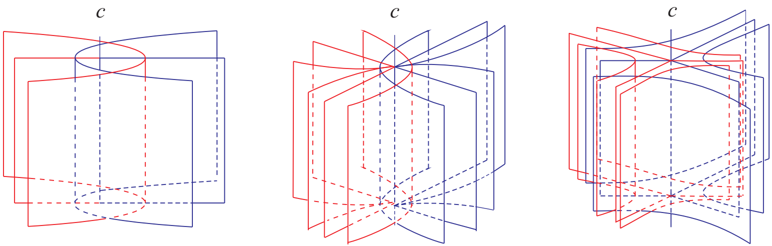


Principal Configurations near a curve of Darbouxian Partially Umbilic Points, [Ga]

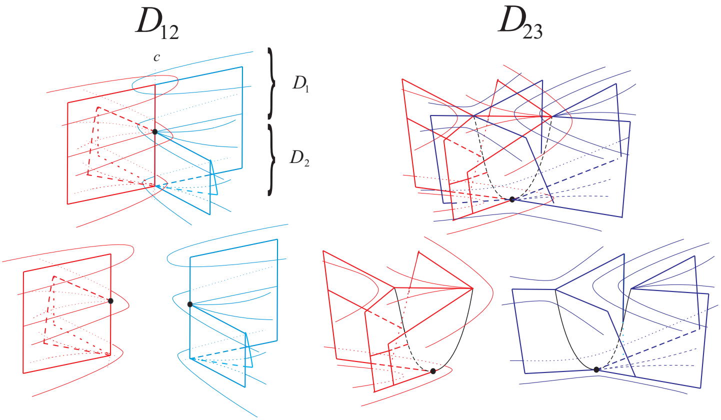


Curve of Darbouxian Partially Umbilic Points.

D_1 D_2 D_3



Generic Transitions of Darbouxian Partially Umbilic Points along a regular partially umbilic curve, [Garcia], 1989.



Darboux for hypersurfaces of $\mathbb{R}^4$. Proofs, 2012, with Lie - Cartan methods.

REMARK on Regularity and Criticality: $L_r(u_1, u_2, u_3) = 0, \ M_r(u_1, u_2, u_3) = 0.$

OUTLINE OF THE ANALYSIS OF SEMI-DARBOUXIAN POINT OF TYPE D_{12}: $cb(b-a) \neq 0$, $\frac{a}{b} = 2$, and $\chi_2 \neq 0$

OUTLINE OF THE ANALYSIS OF SEMI-DARBOUXIAN POINT OF TYPE D_{12}: $cb(b-a) \neq 0$, $\frac{a}{b} = 2$, and $\chi_2 \neq 0$

The Lie-Cartan vector fields
$X = (\mathcal{L}_P, P\mathcal{L}_P, (\mathcal{U} + \mathcal{V}P)\,\mathcal{L}_P, -(\mathcal{L}_{u_1} + P\mathcal{L}_{u_2} + \mathcal{L}_{u_3}(\mathcal{U} + \mathcal{V}P)))$, is given by

OUTLINE OF THE ANALYSIS OF SEMI-DARBOUXIAN POINT OF TYPE D_{12}: $cb(b-a) \neq 0$, $\frac{a}{b} = 2$, and $\chi_2 \neq 0$

The Lie-Cartan vector fields
$X = (\mathcal{L}_P, P\mathcal{L}_P, (\mathcal{U} + \mathcal{V}P)\,\mathcal{L}_P, -(\mathcal{L}_{u_1} + P\mathcal{L}_{u_2} + \mathcal{L}_{u_3}(\mathcal{U} + \mathcal{V}P)))$, is given by

$$\begin{aligned}
\dot{u}_1 &= (-2bu_2 - 2q_{111}u_3)P + (-b)u_1 + cu_2 + (-q_{201} + q_{021})u_3 + O(2)\\
\dot{u}_2 &= PX_1\\
\dot{u}_3 &= \left(\left(\tfrac{q_{111}u_1 + q_{021}v + q_{012}u_3}{k-1} + O(2)\right)P + \tfrac{q_{201}u_1 + q_{111}u_2 + q_{102}u_3}{k-1} + O(2)\right)X_1\\
\dot{P} &= A_3(u_1,u_2,u_3)P^3 + A_2(u_1,u_2,u_3)P^2 + A_1(u_1,u_2,u_3)P + A_0(u_1,u_2,u_3)
\end{aligned} \tag{12}$$

onde

$$
\begin{aligned}
A_3(u_1,u_2,u_3) &= b+\left(C-k^3+\tfrac{q_{111}^2+q_{201}q_{021}}{k-1}\right)u_1+\left(D+3\tfrac{q_{111}q_{021}}{k-1}\right)u_2+\\
&+\left(Q_{121}+\tfrac{2q_{111}q_{012}+q_{102}q_{021}}{k-1}\right)u_3+O(2)\\
A_2(u_1,u_2,u_3) &= c+\left(-D+2B+\tfrac{6q_{111}q_{201}-3q_{111}q_{021}}{k-1}\right)u_1+\\
&+\left(-E+k^3+2C+\tfrac{4q_{111}^2-3q_{021}^2+2q_{201}q_{021}}{k-1}\right)u_2+\\
&+\left(-Q_{031}+2\tfrac{Q_{211}(2q_{201}q_{012}+4q_{102}q_{111}-3q_{012}q_{021})}{k-1}\right)u_3+O(2)\\
A_1(u_1,u_2,u_3) &= \left(-2C+A-k^3+\tfrac{-2q_{201}q_{021}-4q_{111}^2+3q_{201}^2}{k-1}\right)u_1+\\
&+\left(-2D+B+\tfrac{3q_{111}q_{201}-6q_{111}q_{021}}{k-1}\right)u_2+\\
&+\left(-2Q_{121}+Q_{301}+\tfrac{3q_{102}q_{201}-2q_{102}q_{021}-4q_{111}q_{012}}{k-1}\right)u_3+O(2)\\
A_0(u_1,u_2,u_3) &= \left(-B-3\tfrac{q_{111}q_{201}}{k-1}\right)u_1+\left(-C+k^3-\tfrac{2{q_{111}}^2+q_{201}q_{021}}{k-1}\right)u_2+\\
&+\left(-Q_{211}-\tfrac{q_{201}q_{012}+2q_{102}q_{111}}{k-1}\right)u_3+O(2)
\end{aligned}
$$

The singular points of X are given by

$$\begin{cases} \begin{cases} L_r(u_1, u_2, u_3) = 0, \\ M_r(u_1, u_2, u_3) = 0, \end{cases} \quad \text{(Partially Umbilic Points)} \\ A_3(u_1, u_2, u_3)P^3 + A_2(u_1, u_2, u_3)P^2 + A_1(u_1, u_2, u_3)P + A_0(u_1, u_2, u_3) = 0 \end{cases}$$

The singular points of X are given by

$$\begin{cases} \begin{cases} L_r(u_1,u_2,u_3)=0, \\ M_r(u_1,u_2,u_3)=0, \end{cases} \quad \text{(Partially Umbilic Points)} \\ A_3(u_1,u_2,u_3)P^3+A_2(u_1,u_2,u_3)P^2+A_1(u_1,u_2,u_3)P+A_0(u_1,u_2,u_3)=0 \end{cases}$$

- $b(b-a)\neq 0 \Rightarrow$ we can write $u_1=u_1(u_3)$ and $u_2=u_2(u_3)$ in
$$\begin{cases} L_r(u_1,u_2,u_3)=0, \\ M_r(u_1,u_2,u_3)=0, \end{cases}$$

The singular points of X are given by

$$\begin{cases} \begin{cases} L_r(u_1,u_2,u_3)=0, \\ M_r(u_1,u_2,u_3)=0, \end{cases} & \text{(Partially Umbilic Points)} \\ A_3(u_1,u_2,u_3)P^3+A_2(u_1,u_2,u_3)P^2+A_1(u_1,u_2,u_3)P+A_0(u_1,u_2,u_3)=0 \end{cases}$$

- $b(b-a)\neq 0 \Rightarrow$ we can write $u_1=u_1(u_3)$ and $u_2=u_2(u_3)$ in $\begin{cases} L_r(u_1,u_2,u_3)=0, \\ M_r(u_1,u_2,u_3)=0, \end{cases}$
- The discriminant of $A_3(u_1(u_3),u_2(u_3),u_3)P^3+A_2(A_3(u_1(u_3),u_2(u_3),u_3)P^2+A_1(A_3(u_1(u_3),u_2(u_3),u_3)P+A_0(A_3(u_1(u_3),u_2(u_3),u_3)=0$ is given by

$$D(u_3)=\chi_2 u_3+O(2),$$

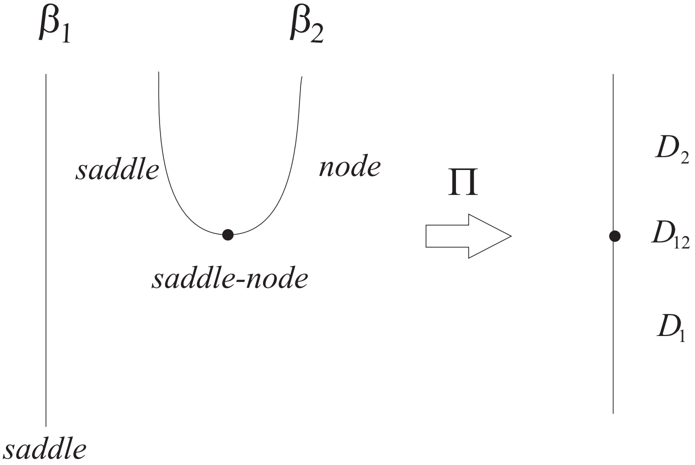


Figure: Equilibrium Curves

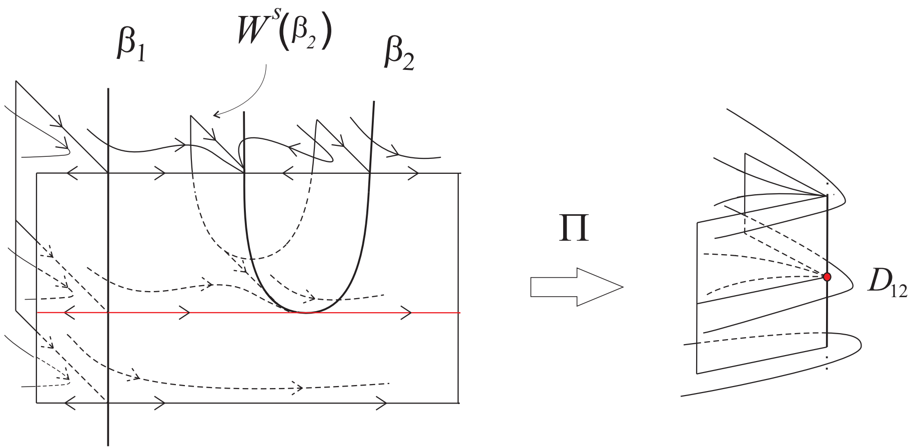


Figure: Lie - Cartan Resolution of D_{12}

Partially Umbilic Points of codimension 1: D^1_1, D^1_2, D^1_3, D^1_{13}, $D^1_{1h,p}$, $D^1_{1h,n}$, D^1_p, D^1_c. There are Eight Generic Types, studied in the Thesis of da Silva, in preparation.

Recap Through Some Historical Landmarks

- G. Monge (1796), definition of principal lines motivated by applications in the transport problem.
- C. Dupin (1818), triply orthogonal system of surfaces.
- G. Darboux (1896), description of lines of curvature near generic umbilics on analytic surfaces
- C. Caratheodory ($\sim$ 1920), question about the minimal number of umbilics on compact and convex surfaces (still an open problem?).
- C. Gutierrez and J. Sotomayor (1982 and after), on qualitative theory of principal curvature lines: stability, bifurcations,....
- R. Garcia (1989) IMPA, Thesis on case $n = 3$, and after....
- Book Garcia - Sotomayor 2009, Survey in ArXiv.

Some References

- G. Darboux, *Leçons Sur la Théorie des Surfaces*, vol. IV. Sur la forme des lignes de Courbure dans la voisinage d'un ombilic, Note 7 , Paris: Gauthier Villars.
- J. Bruce and D. Fidal, *On binary differential equations and umbilics*, Proc. Royal Soc. Edinburgh, 111A,(1989), 147-168.
- R. Garcia, *Linhas de curvatura de hipersuperfícies imersas no espaço* $\mathbb{R}^4$, Pré- Publicao-IMPA, (Thesis), Série **F**, 27, (1989).
- D. Lopes da Silva, *Famlias a* 1 *parmetro de hipersuperfícies imersas em* $\mathbb{R}^4$, Thesis submitted to IME- USP; August, 2012.
- R. Garcia, *Principal Curvature lines near Darbouxian partially umbilic points of hypersurfaces immersed in* $\mathbb{R}^4$, Comp. and Appl. Math., **20**, (2001), pp. 121-148.
- C. Gutierrez and J. Sotomayor, *Structural Stable Configurations of lines of Principal Curvature*, Asterisque, **98-99**, (1982), pp. 185-215.