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THREE LECTURES ON CONTACT GEOMETRY OF MONGE-AMPÈRE EQUATIONS

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Contents

Introduction

The main goal of these lectures is to give a brief introduction to contact geometry of Monge-Ampère equations. Such equations form a subclass of the class of second order partial differential equations.

This subclass is rather wide and contains all linear and quasi-linear equations. On the other hand, it is a minimal class that contains quasilinear equations and that is closed with respect to contact transformations.

This fact was known to Sophus Lie, who applied contact geometry methods to this kind of equations. S. Lie put the classification problem for Monge-Ampère equations with respect to contact pseudogroup. In particular, he put the problem of equivalence of Monge-Ampère equations to the quasilinear and linear forms.

A notion «Monge-Ampère equations» was introduced by Gaston Darboux in his lectures on general theory of surfaces [2, 3, 4]. Equations of the form

$$Av_{xx} + 2Bv_{xy} + Cv_{yy} + D(v_{xx}v_{yy} - v_{xy}^2) + E = 0$$

he calls *Monge-Ampère equations.*

Here A, B, C, D and E are functions on independent variables x, y, unknown function $v = v(x, y)$, and its first derivatives v_x, v_y.

In 1978 Valentin Lychagin noted that the classical Monge-Ampère equations and there multi-dimensional analogues admit effective description in terms of differential forms on the space of 1-jets of smooth functions [23]. His idea was fruitful, and it generated a new approach to Monge-Ampère equations.

The lectures has the following structure.

In the first lecture we give an introducion to geometry of jets space. We define jets of scalar functions on a smooth manifold, introduce the Cartan distribution, contact transformations and contact vector fields.

In the second lecture we consider differential equations as submanifolds of jets manifold. We describe main ideas of Valentin Lychagin and give a short introduction to geometry of the Monge-Ampère equations. Here we follow the papers [23, 24] and the books [18, 25].

The last lecture is devoted to classification of Monge-Ampère equations with two independent variables. Particularly we consider a problem of linearization of Monge-Ampère equations with respect to contact transformations.

Details can be found in [18] and in the original papers (see the bibliography).

Lecture 1

Geometry of jet spaces

1.1 Jets

Let M be an n-dimensional smooth manifold and let $C^\infty(M)$ be the algebra of smooth functions on M. Let $a \in M$ be a point.

A set of all smooth functions on M that vanish at the point a we denote by μ_a, i.e.

$$\mu_a \overset{\text{def}}{=} \{f \in C^\infty(M) \mid f(a) = 0\}.$$

This set is an ideal of the algebra $C^\infty(M)$. Let μ_a^k be the k-th degree of this ideal:

$$\mu_a^k = \left\{ \sum \prod_{i=1}^{k} f_i \,\middle|\, f_i \in \mu_a \right\}$$

In the other words, μ_a^k consists of functions that have zero partial derivatives of order $< k$ at the point a:

$$\mu_a^k = \left\{ f \in C^\infty(M) \,\middle|\, \frac{\partial^{|\sigma|} f}{\partial x^\sigma}(a) = 0,\, 0 \le |\sigma| < k-1 \right\},$$

where $x = (x_1, \ldots, x_n)$ are local coordinates on M, $\sigma = (\sigma_1, \ldots, \sigma_n)$ is a multi-index, $|\sigma| = \sigma_1 + \cdots + \sigma_n$, and

$$\frac{\partial^{|\sigma|} f}{\partial x^\sigma} = \frac{\partial^{|\sigma|} f}{\partial x_1^{\sigma_1} \ldots \partial x_n^{\sigma_n}}.$$

Consider the quotient algebra

$$J_a^k M \overset{\text{def}}{=} C^\infty(M)/\mu_a^{k+1}.$$

Definition 1.1. *Elements of $J_a^k M$ are called* k-jets of functions at the point a.

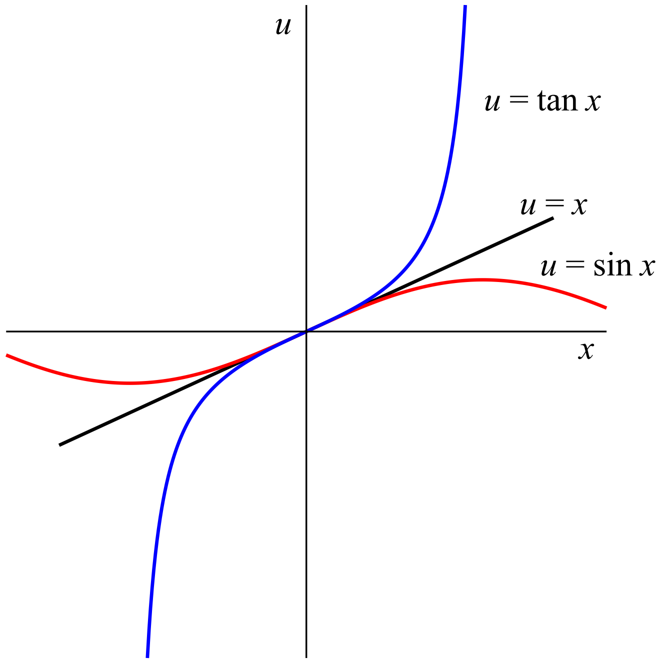

Figure 1.1: 1-jets of functions

The k-jet of a function $f \in C^\infty(M)$ at a point a we denote by $[f]_a^k$, i.e.,

$$[f]_a^k = f \bmod \mu_a^{k+1}.$$

Two functions f and g define the same k-jet at a point a if and only if there corresponding partial derivatives of order $\leq k$ at the point a coincide:

$$f(a) = g(a), \quad \frac{\partial^{|\sigma|} f}{\partial x^\sigma}(a) = \frac{\partial^{|\sigma|} g}{\partial x^\sigma}(a)$$

for $|\sigma| \leq k$. For example (see Fig. 1.1),

$$[x]_0^i = [\sin x]_0^i$$

for $i = 0, 1, 2$, but

$$[x]_0^3 \neq [\sin x]_0^3.$$

The k-jet of the function $(x-a)^m$ at the point a is zero if $k < m$:

$$[(x-a)^m]_a^k = [0]_a^k. \tag{1.1}$$

The polynomial

$$f(a) + \sum_{|\sigma| \leq k} \frac{1}{\sigma!} \frac{\partial f^{|\sigma|}}{\partial x^\sigma}(a)(x-a)^k$$

we shall consider as a representative of k-jet $[f]_a^k$ in the given coordinates.

Let us introduce a structure of a vector space on $J_a^k M$. Define a summa of two jets and a product a jet by a real number by the following formulas:

$$[f]_a^k + [g]_a^k \stackrel{\text{def}}{=} [f+g]_a^k, \quad \lambda[f]_a^k \stackrel{\text{def}}{=} [\lambda f]_a^k.$$

Theorem 1.1. *The k-jets*

$$[1]_a^k, \quad [x-a]_a^k, \ldots, \frac{1}{\sigma!}[(x-a)^\sigma]_a^k \quad (|\sigma| \leq k).$$

form a basis of the vector space $J_a^k M$.

Proof. Due to the Taylor formula, we have:

$$f(x) = f(a) + \sum_{|\sigma| \leq k} \frac{1}{\sigma!} \frac{\partial f^{|\sigma|}}{\partial x^\sigma}(a)(x-a)^k + o((x-a)^k).$$

and

$$[f]_a^k = f(a)[1]_a^k + \sum_{|\sigma| \leq k} \frac{\partial f^{|\sigma|}}{\partial x^\sigma}(a) \frac{[(x-a)^\sigma]_a^k}{\sigma!}.$$

□

Exercise 1. *Prove that*

$$\dim J_a^k M = \binom{k+n}{k}.$$

Definition 1.2. *The union of all k-jets, i.e.,*

$$J^1 M \stackrel{\text{def}}{=} \bigcup_{a \in M} J_a^1 M$$

is called a space of k-jets *of functions on M.*

On this space we define a structure of smooth manifold with local coordinates

$$x_1, \ldots, x_n, u, p_\sigma, \quad (|\sigma| \leq k)$$

where

$$x_i([f]_a^k) = x_i(a), \quad u([f]_a^k) = f(a), \quad p_\sigma([f]_a^k) = \frac{\partial^{|\sigma|} f}{\partial x^\sigma}(a) \quad (|\sigma| \leq k).$$

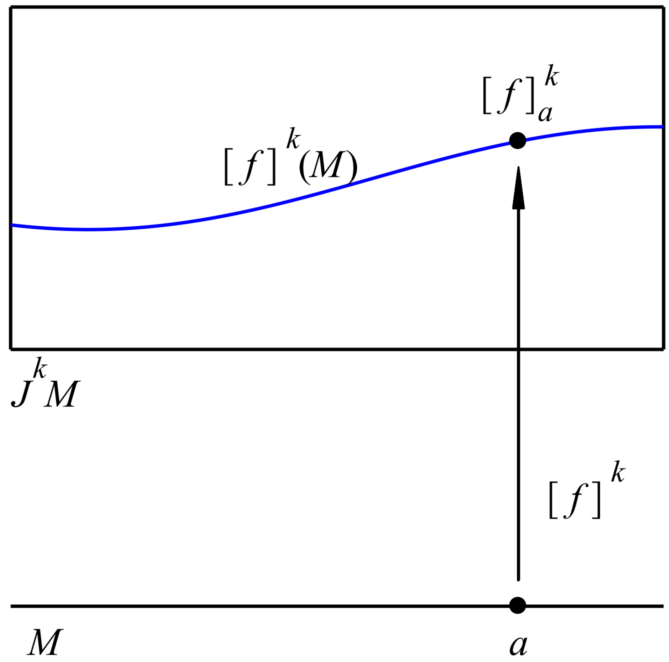

Figure 1.2: The map $[f]^k$

Corollary 1.

$$\dim J^k M = n + \binom{k+n}{k}.$$

With functions $f \in C^\infty(M)$ we associate maps

$$[f]^k : M \to J^k M$$

where

$$[f]^k(a) = [f]^k_a.$$

Definition 1.3. *The map $[f]^k$ is called the k-*prolongation *of the function f.*

The image of M is a smooth submanifold

$$L^k_f = [f]^k(M) \subset J^k M$$

*which we call k-*graph *of the function f (see Fig. 1.2).*

Exercise 2. *Prove that the bundles*

$$\pi_k\colon J^k M \to M, \qquad \pi_k\colon [f]^k_a \mapsto a$$

are a vector bundles.

1.2 Cartan distribution

Consider the following problem: describe a class of all n-dimensional submanifolds of J^kM that are k-graphs of functions.

Note that if $L \subset J^kM$ is a k-graph of a function, then the projection $\pi : L \to M$ is a diffeomophism. This means that local coordinates on M can be viewed as local coordinates on L.

On the other hand, a submanifold

$$L = \{u = f(x), p_\sigma = g_\sigma(x)\} \subset J^kM,$$

is a k-graph of a function if and only if

$$g_\sigma = \frac{\partial^{|\sigma|} f}{\partial x^\sigma}, \qquad |\sigma| \leq k.$$

We shall describe such submanifolds in a coordinate-free form. For this goal let's introduce the so-called *Cartan distribution.*

At first,recall some facts of distributions theory.

A k-dimensional distribution P on a smooth manifold N is a «smooth» field

$$P\colon a \in N \mapsto P(a) \subset T_aN$$

of k-dimensional subspaces of the tangent spaces.

There are two main ways to say that P is a smooth field, and to define a distribution: by vector fields or by differential 1-forms.

By vector fields. Let $X_1, \ldots, X_k$ be such vector fields on N that $P(a)$ is span by the tangent vectors $X_{1,a}, \ldots, X_{k,a}$:

$$P(a) = \mathrm{Span}(X_{1,a}, \ldots, X_{k,a}).$$

In this case we write

$$P = \langle X_1, \ldots, X_k\rangle.$$

We say that a vector field X *belongs* to P if $X_a \in P(a)$ for any point $a \in N$. Then smoothness of P means that there are local bases for P consisting of vector fields that belong to P.

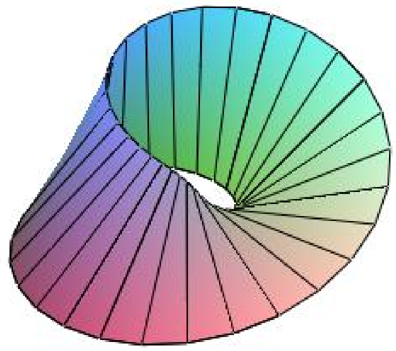

Figure 1.3: One-dimensional distribution on Mobius strip

We denote by $D(P)$ the set of all vector fields that belong to P.

By differential 1-forms. Suppose that there are $n-k$ differential 1-forms $\omega_1, \dots, \omega_{n-k}$ ($n = \dim N$) such that a subspace $P(a)$ is the intersection of kernels of these forms:

$$P(a) = \bigcap_{i=1}^{n-k} \ker \omega_{i,a}.$$

In this case we write

$$P = \langle \omega_1, \dots, \omega_{n-k} \rangle.$$

The smoothness of P means that locally the distribution can be defined by a set of differential 1-forms.

Remark 1. *For general smooth manifolds distributions can be defined by vector fields or by differential forms only locally. For example, the one-dimensional distribution P on the Mobius strip (see Fig. 1.3) cannot be defined by a vector field globally: each vector field which belongs to this distributions has a singular point.*

A submanifold $L \subset N$ is said to be *integral* for the distribution P if

$$T_a L \subset P(a)$$

for any point $a \in L$.

Return to the Cartan distribution.

Let $\theta \in J^k M$ and $a = \pi_k(\theta)$. Then the subspace

$$\mathcal{C}^k(\theta) = \mathrm{Span} \bigcup_{f \in C^\infty(M), [f]_a^k = \theta} T_\theta L_f^k$$

is called *the Cartan subspace.*

The distribution

$$\mathcal{C}^k : J^k M \ni \theta \mapsto \mathcal{C}^k(\theta) \subset T_\theta J^k M$$

is called the *Cartan distribution.*

By the construction, k-graphs of functions are integral manifolds of the Cartan distribution.

Note that not any integral manifold $L \subset J^k M$ of the Cartan distribution is a k-graph of a function. This is true if and only the projection $\pi : L \to M$ is a diffeomophism.

Theorem 1.2. *An n-dimensional smooth manifold L is a k-graph of a smooth function on M if and only if*

- *L is an integral manifold of the Cartan distribution,*
- *the projection $\pi : L \to M$ is a diffeomophism.*

Example 1. *Consider a case when $M = \mathbb{R}$. Then $J^1\mathbb{R} = \mathbb{R}^3$ and coordinates on $J^1\mathbb{R}$ are such functins x, u, p, that*

$$x([f]_a^1) = a, \quad u([f]_a^1) = f(a), \quad p([f]_a^1) = f'(a).$$

Then the curve

$$L_f^1 = \{u = f(x), p = f'(x)\}$$

is a 1-graph of the function f.

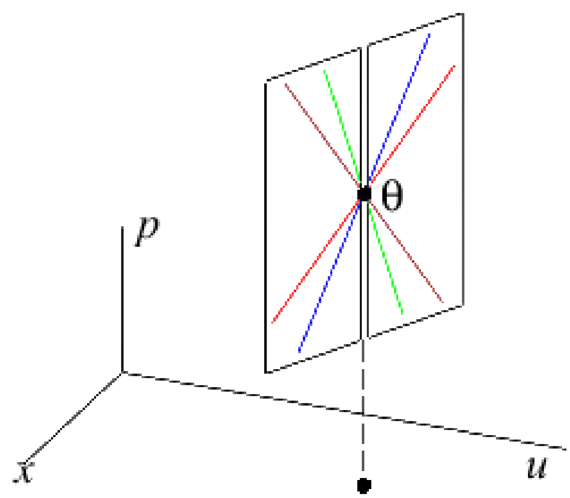

Figure 1.4: Cartan subspace for $n = 1$

A tangent line $T_\theta L_f^1$ *at a point* $\theta = [f]_a^1$ *is generated by the vector*

$$\left.\frac{\partial}{\partial x}\right|_\theta + f'(a)\left.\frac{\partial}{\partial u}\right|_\theta + f''(a)\left.\frac{\partial}{\partial p}\right|_\theta$$

All possible tangent lines $T_\theta L_f^1$ *are lines fin the plane*

$$\{du - p_\theta dx = 0\} \subset T_\theta(J^1\mathbb{R}) \tag{1.2}$$

without the vertical line (see picture 1.4). Here a, u_θ, p_θ *are coordinates of the point* θ*. Therefore they span is plane (1.2).*

So, in this case the Cartan space $\mathcal{C}(\theta)$ *is a plane which is generated by the vectors*

$$\left.\frac{\partial}{\partial x}\right|_\theta + p_\theta\left.\frac{\partial}{\partial u}\right|_\theta, \qquad \left.\frac{\partial}{\partial p}\right|_\theta.$$

Therefore, the Cartan distribution is generated by the corresponding vec-

tor fields:

$$\frac{\partial}{\partial x} + p\frac{\partial}{\partial u}, \quad \frac{\partial}{\partial p}$$

or, equivalently, by the differential 1-form

$$\omega_0 = du - pdx.$$

This form is called Cartan form.

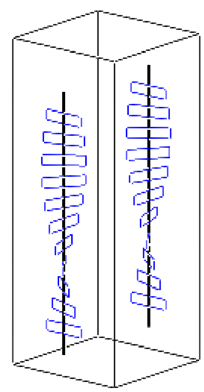

Figure 1.5: Cartan distribution for $n = 1$

1.3 Contact structure on J^1M

Consider the case when $M = \mathbb{R}^n$. Then $J^1\mathbb{R}^n = \mathbb{R}^{2n+1}$ and coordinates on $J^1\mathbb{R}$ are

$$x_1, \ldots, u, p_1, \ldots, p_n$$

where

$$x_i([f]^1_a) = a_i, \quad u([f]^1_a) = f(a), \quad p_i([f]^1_a) = \frac{\partial f}{\partial x_i}(a).$$

The Cartan distribution is generated by the vector fields

$$\frac{\partial}{\partial x_i} + p_i\frac{\partial}{\partial u}, \qquad \frac{\partial}{\partial p_i} \qquad (i = 1, \ldots, n),$$

or by the differential 1-form

$$\omega_0 = du - p_1 dx_1 - \cdots - p_n dx_n. \tag{1.3}$$

Recall that a contact structure on odd-dimensional smooth manifold is a maximal non-integrable distribution of codimensional 1. This means the following.

Let $P = \langle \omega \rangle$ be a $2n$-dimensional distribution on $2n+1$-dimensional smooth manifold N. This distribution is called a *contact structure* if $\omega \wedge (d\omega)^n \neq 0$.

The following theorem gives a canonical representation for contact structures.

Theorem 1.3 (G. Darboux)**.** *Let $P = \langle \omega \rangle$ be a contact structure. There exist local coordinates $x_1, \ldots, x_n, y_1, \ldots, y_n, z$ on N such that*

$$\omega = dz - y_1 dx_1 - \cdots - y_n dx_1.$$

Since (1.3), we have

$$d\omega_0 = dq_1 \wedge dp_1 + \cdots + dq_n \wedge dp_n. \tag{1.4}$$

Therefore,

$$\omega_0 \wedge (d\omega_0)^n \neq 0$$

and we see that the Cartan distribution $\mathcal{C}$ defines the contact structure on J^1M.

Let θ be a point of J^1M. A restriction of $d\omega_0$ on the Cartan space $\mathcal{C}(\theta)$ defines a symplectic structure. We denote it by Ω_θ, i.e.,

$$\Omega_\theta = d\omega_0|_{\mathcal{C}(\theta)}.$$

1.4 Contact transformations

Definition 1.4. *A diffeomophism $\varphi : J^1M \to J^1M$ is called* contact *if it preserves the Cartan distribution.*

In other words, contact diffeomorphisms are symmetries of the Cartan distribution.

This means that

$$\varphi^*(\omega_0) = \lambda_\varphi \omega_0$$

for a smooth function λ_φ on J^1M.

The following transformations are contact:

1. Translations:

$$(x, u, p) \longmapsto (x + \alpha, u + \beta, p),$$

   where α, β are constant.

2. Scale transformations

$$(x, u, p) \longmapsto (e^\alpha x, e^\beta u, e^{(\beta-\alpha)} p).$$

3. The Legendre transformation:

$$(x, u, p) \longmapsto (p, u - xp, -x)$$

4. The Euler (or partial Legendre's) transformation:

$$(x_1, x_2, u, p_1, p_2) \longmapsto (p_1, x_2, u - p_1 x_1, -x_1, p_2)$$

5. Shifts:

$$(x, u, p) \mapsto \left(x, u + h, p + \frac{\partial h}{\partial x}\right),$$

   where $h = h(x)$.

Important class of contact transformations can be obtained as prolongations of transformations of the space J^0M.

Diffeomophism of the 0-jets space are called *point transformations.*

Let

$$\varphi : (x, u) \mapsto (X(x, u), U(x, u))$$

be a point transformation.

Definition 1.5. *The transformation*

$$\varphi^{(1)} : (x, u, p) \mapsto (X(x, u), U(x, u), P(x, u, p))$$

is called the first prolongation *or* the prolongation *of the transformation* φ *to the space* J^1M *if it preserves the Cartan distribution.*

Exercise 3. *Prove that for the first prolongation*

$$P = \frac{\frac{dY}{dx}}{\frac{dX}{dx}},$$

where

$$\frac{d}{dx} = \frac{\partial}{\partial x} + p\frac{\partial}{\partial u}.$$

1.5 Contact vector fields

Infinitesimal analogies of contact transformations are contact vector fields.

Let X be a vector field on J^1M and let $\{\varphi_t\}$ be a local translation group of X.

Definition 1.6. *The vector field X is called* contact *if its translation group consists of contact transformations, i.e.*

$$\varphi_t^*(\omega_0) = \lambda_t \omega_0 \tag{1.5}$$

for some function λ_t on J^1M.

After differentiating both parts of (1.5) by t at $t = 0$, we get:

$$\left.\frac{d}{dt}\right|_{t=0} \varphi_t^*(\omega_0) = \left.\frac{d\lambda}{dt}\right|_{t=0} \omega_0. \tag{1.6}$$

The left part of the equation is the Lie derivative $L_X(\omega_0)$ of the Cartan form by the vector field X. Multiplying both parts of the last equation by the form ω_0 we get:

$$L_X(\omega_0) \wedge \omega_0 = 0. \tag{1.7}$$

Find a coordinate representation of contact vector fields. For simplify our calculations we suppose that $n = 1$. Let x, u, p be coordinates on J^1M.

Let

$$X = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial u} + c\frac{\partial}{\partial p}$$

be a contact vector field. Here a, b, c are smooth functions on J^1M. Using the formula

$$L_X = d \circ \iota_X + \iota_X \circ d,$$

we get

$$L_X(\omega_0) = db - cdx - pda.$$

Formula (1.7) gives two differential equations on the components of X :

$$p\frac{\partial a}{\partial p} - \frac{\partial b}{\partial p} = 0,$$
$$\frac{\partial b}{\partial x} - p\frac{\partial a}{\partial x} - c + p\left(\frac{\partial b}{\partial u} - p\frac{\partial a}{\partial u}\right) = 0.$$

Let us put

$$f = b - pa.$$

Then

$$a = -\frac{\partial f}{\partial p}, \quad b = f - p\frac{\partial f}{\partial p}, \quad c = \frac{\partial f}{\partial x} + p\frac{\partial f}{\partial u}$$

and any contact vector field determines by a smooth function f and has the following form:

$$X_f = -\frac{\partial f}{\partial p}\frac{\partial}{\partial x} + \left(f - p\frac{\partial f}{\partial p}\right)\frac{\partial}{\partial u} + \left(\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial u}\right)\frac{\partial}{\partial p}.$$

Note that

$$\omega_0(X_f) = f.$$

For arbitrary n a contact vector field has the form

$$X_f = -\sum_{i=1}^{n}\frac{\partial f}{\partial p_i}\frac{\partial}{\partial x_i} + \left(f - \sum_{i=1}^{n} p_i\frac{\partial f}{\partial p_i}\right)\frac{\partial}{\partial u} + \sum_{i=1}^{n}\left(\frac{\partial f}{\partial x_i} + p_i\frac{\partial f}{\partial u}\right)\frac{\partial}{\partial p_i}$$

for some function f.

1.6 Lie transformations

Lie transformations are generalization of contact transformations.

Definition 1.7. *A diffeomophism $\varphi : J^kM \to J^kM$ is called a Lie transformation if it preserves the Cartan distribution $\mathcal{C}^k$.*

For $k = 1$ Lie transformations are contact ones.

Each Lie transformation on J^kM can be prolonged to the space $J^{k+1}M$.

Theorem 1.4. *Any Lie transformation on J^kM is is a $(k-1)$-th prolongation of a contact transformation.*

Lecture 2

Differential equations

2.1 Multivalued solutions

Consider a scalar k-th order differential equation on a smooth manifold M

$$F\left(x, v, \frac{\partial^{|\sigma|} v}{\partial x^\sigma}\right) = 0, \qquad |\sigma| \leq k. \tag{2.1}$$

This equation can be viewed as a hypersurface

$$\mathcal{E} = \{F(x, u, p_\sigma) = 0\} \subset J^k M$$

in the space of k-jets. This hypersurface we call an *equation* too.

Theorem 2.1. *A function $v = h(x)$ is a solution of equation (2.1) if and only if its k-graph lies on the hypersurface $\mathcal{E}$.*

Proof. A function $v = h(x)$ is a solution of (2.1) if and only if

$$F\left(x, h(x), \frac{\partial^{|\sigma|} h(x)}{\partial x^\sigma}\right) \equiv 0.$$

This means that $L_h^k \subset \mathcal{E}$. □

Definition 2.1. *An n-dimensional integral manifold L of the Cartan distribution is called a multivalued solution of equation $\mathcal{E}$ if $L \subset \mathcal{E}$.*

Example 2. *The ordinary differential equation*

$$(y')^2 = x \tag{2.2}$$

generates the following hypersurface in $J^1\mathbb{R}$:

$$\mathcal{E} = \{p^2 - x = 0\}.$$

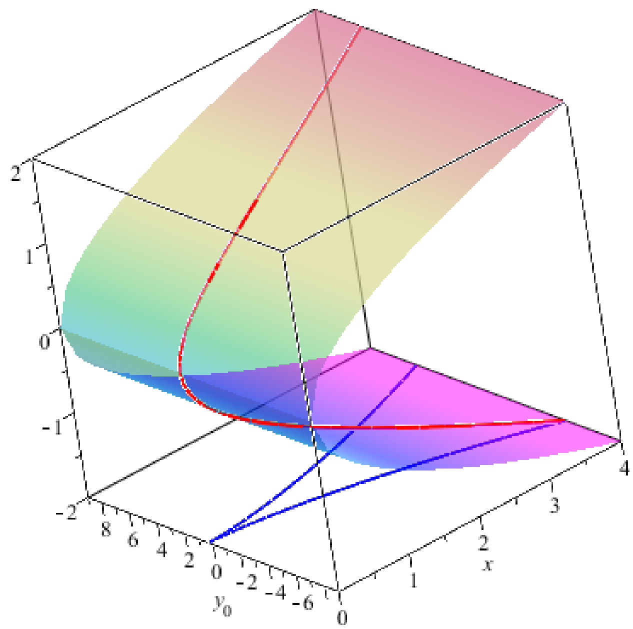

Figure 2.1: Equation $p^2 = x$ and its multivalued solution

The curve

$$L = \left\{ x = t^2, \quad u = \frac{2}{3}t^3, \quad p = t \right\}$$

(see Fig. 2.2) is a multivalued solution of the equation.

Exercise 4. *Construct multivalued solutions of the following equation:*

$$(y')^3 - 3y' - y + 1 = 0.$$

Example 3. *The Lissajou figure*

$$u = \cos(at), x = \sin(bt)$$

corresponds to a multivalued solution of the differential equation

$$(1 - x^2)y'' + xy' + \lambda y = 0$$

with $\lambda = a^2/b^2$.

Example 4. *Spheres with radius 1 in the space* $\mathbb{R}^3(x, y, u)$ *are projections of multivalued solutions of the equation*

$$\frac{v_{xx}v_{yy} - v_{xy}^2}{(1 + v_x^2 + v_y^2)^2} = 1. \tag{2.3}$$

into $J^0\mathbb{R}^2$.

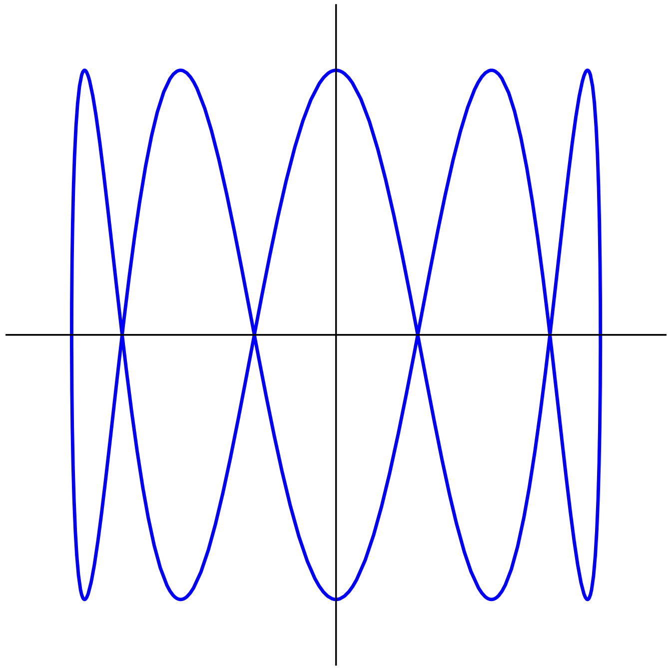

Figure 2.2: The Lissajou figure

2.2 Lychagin's approach to Monge-Ampère equations

Let M be an n-dimensional smooth manifold and let J^1M be the manifold of 1-jets of smooth functions on M.

The manifold J^1M is equipped with Cartan's distribution

$$\mathcal{C} : a \in J^1M \mapsto \mathcal{C}(a) \subset T_a(J^1M)$$

given by the differential 1-form ω_0.

In the canonical local coordinates on J^1M the Cartan form

$$\omega_0 = du - p_1dx_1 - \cdots - p_ndx_n.$$

A main idea of V. Lychagin is the following [23].

With any differential n-form ω on J^1M we can associate a differential operator

$$\Delta_\omega : C^\infty(M) \to \Omega^n(M),$$

which acts by the following way:

$$\Delta_\omega(v) = \omega|_{L_v^1}.$$

Here L_v^1 is the 1-graph of the function v.

This construction does not cover all nonlinear second order differential operators, but only a certain subclass of them.

Example 5. *Let*

$$\omega = 3p^2 dp - dx \tag{2.4}$$

be the differential 1-form on $J^1(\mathbb{R})$. *The corresponding operator* Δ_ω *acts as*

$$\Delta_\omega(v) = \big(3(v')^2 v'' - 1\big)\, dx. \tag{2.5}$$

Indeed,

$$\Delta_\omega(v) = (3p^2 dp - dx)|_{j^1(v)(M)} = 3(v')^2 d\,(v') - dx = \big(3(v')^2 v'' - 1\big)\, dx.$$

Example 6. *The differential 2-form*

$$\omega = dp_1 \wedge dp_2$$

on $J^1\mathbb{R}^2$ *corresponds to operator*

$$\begin{aligned}\Delta_\omega(v) &= d\,(v_{x_1}) \wedge d\,(v_{x_2})\\ &= (v_{x_1x_1}dx_1 + v_{x_1x_2}dx_2) \wedge (v_{x_2x_1}dx_1 + v_{x_2x_2}dx_2)\\ &= \big(v_{x_1x_1}v_{x_2x_2} - v_{x_1x_2}^2\big)\, dx_1 \wedge dx_2\\ &= (\det \operatorname{Hess} v)\, dx_1 \wedge dx_2.\end{aligned}$$

where

$$\operatorname{Hess} v = \det \left\| \begin{matrix} v_{x_1x_1} & v_{x_1x_2} \\ v_{x_1x_2} & v_{x_2x_2} \end{matrix} \right\|$$

is the Hessian of the function v.

Thus,

$$\Delta_\omega(v) = (\operatorname{Hess} v)\, dx_1 \wedge dx_2. \tag{2.6}$$

Example 7. *The differential* 3*-form*

$$\omega = dx_1 \wedge dp_1 \wedge dp_3 \tag{2.7}$$

on $J^1\mathbb{R}^3$ *produces the following differential operator :*

$$\Delta_\omega(v) = dx_1 \wedge d\,(v_{x_1}) \wedge d\,(v_{x_3}) = (v_{x_1x_2}v_{x_3x_3} - v_{x_1x_3}v_{x_2x_3})\, dx_1 \wedge dx_2 \wedge dx_3.$$

Example 8. *The differential 2-form*

$$\omega = dp_1 \wedge dx_2 - dp_2 \wedge dx_1$$

on $J^1\mathbb{R}^2$ represents the 2-dimensional Laplace operator

$$\Delta_\omega(v) = (v_{x_1x_1} + v_{x_2x_2})\, dx_1 \wedge dx_2.$$

The constructed operator Δ_ω and the equation

$$\mathcal{E}_\omega = \{\Delta_\omega(v) = 0\} \subset J^2M$$

are called the *Monge-Ampère operator* and the *Monge-Ampère equation*, respectively.

The following observation justifies this definition: being written in local canonical contact coordinates on J^1M, the operators Δ_ω have the same type of nonlinearity as the Monge-Ampère operators. Namely, the nonlinearity involves the determinant of the Hesse matrix and its minors. For instance, in the case $n = 2$ we get classical Monge-Ampère equations

$$Av_{xx} + 2Bv_{xy} + Cv_{yy} + D(v_{xx}v_{yy} - v_{xy}^2) + E = 0. \tag{2.8}$$

An advantage of this approach is the reduction of the order of the jet space: we use the simpler space J^1M instead of the space J^2M where Monge-Ampère equations should be *ad hoc* as second-order partial differential equations [29].

The two next examples show that the constructed map «differential n-forms» $\to$ «differential operators» is not a 1-to-1 map: it has a huge kernel.

Example 9. *Two differential 2-forms*

$$\omega = dx_1 \wedge du \qquad \text{and} \qquad \varpi = p_2 dx_1 \wedge dx_2$$

on $J^1\mathbb{R}^2$ generate the same operator:

$$\begin{aligned} \Delta_\omega(v) &= dx_1 \wedge (v_{x_1}dx_1 + v_{x_2}dx_2) = v_{x_2}\, dx_1 \wedge dx_2, \\ \Delta_\varpi(v) &= v_{x_2}\, dx_1 \wedge dx_2. \end{aligned}$$

Example 10. *Any differential n-form*

$$\omega = \omega_0 \wedge \alpha + d\omega_0 \wedge \beta \tag{2.9}$$

on J^1M, where $\alpha \in \Omega^{n-1}\left(J^1M\right)$, $\beta \in \Omega^{n-2}\left(J^1M\right)$ and ω_0 is the Cartan form, gives the zero operator.

This kernel consists of differential forms that vanish on any integral manifold of the Cartan distribution. Due to Lepage's theorem [24], all such forms have form (2.9). They generate a graded ideal

$$I^* = \oplus_{s \geq 0} I^s,$$

$I^s \subset \Omega^s(J^1M)$ of the exterior algebra $\Omega^*(J^1M)$.

Definition 2.2. *Elements of the quotient module $\Omega^s_\varepsilon(J^1M) = \Omega^s(J^1M)/I^s$ are called* effective *s-forms.*

By ω_ε denote the corresponding to ω class in $\Omega^s_\varepsilon(J^1M)$: $\omega_\varepsilon = \omega + I^s$.

Suppose that $n = 2$ and find a coordinate representation of effective 2-forms.

For any element of the factor module Ω^2_ε, one can choose a unique representative $\omega \in \Omega^2(J^1M)$ such that $X_1 \rfloor \omega = 0$ and $\omega \wedge d\omega_0 = 0$. Here X_1 is a contact vector fields with generating function 1. In the local Darboux coordinates such representatives have the form

$$\begin{aligned} \omega =& E dq_1 \wedge dq_2 + B\left(dq_1 \wedge dp_1 - dq_2 \wedge dp_2\right) + \\ & C dq_1 \wedge dp_2 - A dq_2 \wedge dp_1 + D dp_1 \wedge dp_2, \end{aligned} \tag{2.10}$$

where A, B, C, D and E are smooth functions on J^1M.

We identify effective forms as elements of the factor module $\Omega^2_\varepsilon(J^1M)$ with differential forms of type (2.10) and also call such differential forms effective.

Form (2.10) corresponds to equation (2.8).

2.3 Contact transformations of Monge-Ampère equations

Let $\varphi : J^1M \to J^1M$ be a contact transformation. This transformation preserves the ideal I^*: $\varphi^*(I^s) = I^s$, and therefore determines a map of effective forms:

$$\varphi^* : \omega + I^s \mapsto \varphi^*(\omega) + I^*.$$

But φ, in general, does not preserves the Cartan form ω_0, and therefore, does not acts directly on representatives of the effective forms. Thus we shall define the action φ on ω by taking the effective part $\varphi^*(\omega)_\varepsilon$.

Note that, for any contact transformation and any integral manifold of the Cartan distribution $L \subset J^1M$ we have:

$$\varphi^*\left(\omega\right)_\varepsilon|_L = \omega|_{\varphi(L)},$$

Hence, if L is a multivalued solution of the equation $\mathcal{E}_{\varphi^*(\omega)_\varepsilon}$, then $\varphi(L)$ is a multivalued solution of the equation $\mathcal{E}_\omega$.

Definition 2.3. *Two Monge-Ampère equations $\mathcal{E}_\omega$ and $\mathcal{E}_\varpi$ are* contact equivalent *if and only if there exists a contact transformation φ and a nonvanishing function $\lambda_\varphi \in C^\infty(J^1M)$ such that*

$$\omega = \lambda_\varphi \varphi^*(\varpi)_\varepsilon.$$

As a corollary of our interpretation of Monge-Ampère equations we get the following theorem.

Theorem 2.2 (Sophus Lie)**.** *The class of Monge-Ampère equations is closed with respect to contact transformations.*

Example 11. *The Legendre transformation*

$$\varphi : (x,\ u,\ p) \mapsto (p,\ u - px,\ -x)$$

transforms the form (2.4) to the form

$$\varphi^*(\omega) = 3x^2dx - dp.$$

Then the non-linear equation

$$3(v')^2v'' - 1 = 0$$

turns to the following linear equation:

$$v'' = 3x^2.$$

Example 12. *The Von Karman equation*

$$v_{x_1}v_{x_1x_1} - v_{x_2x_2} = 0 \tag{2.11}$$

becomes linear equation

$$x_1v_{x_2x_2} + v_{x_1x_1} = 0 \tag{2.12}$$

after Legendre transformation (2.14). The last equation is known as the Triccomi *equation.*

Example 13 (The Monge-Ampere Equation). *This equation has the form*

$$\operatorname{Hess} v = 1$$

and generated by the effective differential 2-form

$$\omega = dp_1 \wedge dp_2 - dx_1 \wedge dx_2.$$

After the Euler transformation

$$\varphi : (x_1,\ x_2,\ u,\ p_1,\ p_2) \mapsto (p_1,\ x_2,\ u - p_1 x_1,\ -x_1,\ p_2,).$$

it becomes

$$\omega = dx_2 \wedge dp_1 - dx_1 \wedge dp_2,$$

and corresponds to the Laplace equation

$$v_{x_1x_1} + v_{x_2x_2} = 0. \tag{2.13}$$

Example 14 (Quasilinear equations). *Let's consider a quasi-linear equation of the form:*

$$A(v_x, v_y)\, v_{xx} + 2B(v_x, v_y)\, v_{xy} + C(v_x, v_y)\, v_{yy} = 0.$$

This equation is represented by the following effective form

$$\omega = B(p_1, p_2)(dx_1 \wedge dp_1 - dx_2 \wedge dp_2) + C(p_1, p_2)\, dx_1 \wedge dp_2 - A(p_1, p_2)\, dx_2 \wedge dp_1.$$

After the Legendre transformation

$$\varphi : (x_1,\ x_2,\ u,\ p_1,\ p_2) \mapsto (p_1,\ p_2,\ u - p_1 x_1 - p_2 x_2,\ -x_1,\ -x_2,) \tag{2.14}$$

we get the following effective form

$$\begin{aligned}\varphi^*(\omega) = {} & B(-x_1, -x_2)(dx_1 \wedge dp_1 - dx_2 \wedge dp_2) + \\ & - A(-x_1, -x_2)\, dx_1 \wedge dp_2 + C(-x_1, -x_2)\, dx_2 \wedge dp_1,\end{aligned}$$

which corresponds to linear equation:

$$-A(-x_1, -x_2)\, v_{x_2x_2} + 2B(-x_1, -x_2)\, v_{x_1x_2} - C(-x_1, -x_2)\, v_{x_1x_1} = 0.$$

Lecture 3

Classification of Monge-Ampère equations

3.1 The Sophus Lie problem

The problem of equivalence and classification of the Monge-Ampère equations with two independent variables x and y goes back to Sophus Lie's papers from the 1870s and 1880s [20, 21, 22].

Lie have raised the following problem.

Find equivalence classes of nonlinear second-order differential equations with respect to the group of contact transformations.

The important steps in a solution of this problem were made by Darboux [2, 3, 4] and Goursat [**?**, **?**, **?**], who had basically treated the hyperbolic Monge-Ampère equations.

Lie himself had found conditions to transform a Monge-Ampère equation to a quasilinear one and to some linear equation with constant coefficients. But a complete proof of Lie's theorems had never been published. A problem of reducibility of hyperbolic and elliptic Monge-Ampère equations, whose coefficients do not depend on the variable v (they call such equations *symplectic*), to the equations with constant coefficients was solved by Lychagin and Rubtsov in 1983 [26].

3.2 Geometry of Monge-Ampère equations on two-dimensional manifolds

The Monge-Ampère equations on two-dimensional manifolds possess remarkable geometric structures. We consider them for the general and symplectic equations separately.

In what follows, we suppose that $n = 2$ and consider classical Monge-Ampère equations (2.8) only.

With any effective differential 2-form ω one can associate a smooth function $\mathrm{Pf}(\omega)$ on J^1M as follows [26]:

$$\mathrm{Pf}(\omega)\Omega \wedge \Omega = \omega \wedge \omega. \tag{3.1}$$

This function is called *Pfaffian* of ω.

If ω has the form (2.10) then

$$\mathrm{Pf}(\omega) = DE - AC + B^2.$$

We say that the Monge-Ampère equation E_ω is *hyperbolic, elliptic* or *parabolic* at a domain $\mathcal{D} \subset J^1M$ if the function $\mathrm{Pf}(\omega)$ is negative, positive or zero at each point of $\mathcal{D}$, respectively.

If the Pfaffian changes a sign in some points of $\mathcal{D}$, then the equation E_ω is called a *mixed type* equation.

The hyperbolic and elliptic equations are called *nondegenerate.*

Define a non-holonomic field endomorphisms A_ω which is associated with effective form ω.

Since the 2-form Ω_a is non-degenerated on the Cartan distribution $\mathcal{C}(a)$, the operator A_{ω_a} is uniquely defined by the following formula [26]:

$$A_{\omega_a}X_a \rfloor\ \Omega_a = X_a \rfloor \omega_a. \tag{3.2}$$

Here $X_a \in \mathcal{C}(a)$.

To find a coordinate representation of the operator A_ω consider the following basis of the module $D(\mathcal{C})$:

$$\frac{d}{dq_1} \overset{\text{def}}{=} \frac{\partial}{\partial q_1} + p_1\frac{\partial}{\partial u}, \quad \frac{d}{dq_2} \overset{\text{def}}{=} \frac{\partial}{\partial q_2} + p_2\frac{\partial}{\partial u}, \quad \frac{\partial}{\partial p_1}, \quad \frac{\partial}{\partial p_2} \tag{3.3}$$

Then, in this basis, one has the form

$$A_\omega = \begin{Vmatrix} B & -A & 0 & -D \\ C & -B & D & 0 \\ 0 & E & B & C \\ -E & 0 & -A & -B \end{Vmatrix} \tag{3.4}$$

in basis (3.3).

The square of operator A_ω is scalar and

$$A_\omega^2 + \mathrm{Pf}(\omega) = 0. \tag{3.5}$$

Remark that the operator A_ω is symmetric with respect to Ω, i.e.,

$$\Omega(A_\omega X, Y) = \Omega(X, A_\omega Y) \tag{3.6}$$

for any $X, Y \in D(\mathcal{C})$ [26].

It is clear that effective forms ω and $h\omega$, where h is any nonvanishing function, define the same equation.

Therefore, for a non-degenerate equation E_ω the form ω can be normed in such a way that

$$|\,\mathrm{Pf}(\omega)| = 1.$$

Then, due to (3.5), the hyperbolic and elliptic equations generate a product structure

$$A_{\omega,a} = 1$$

and a complex structure

$$A_{\omega,a} = -1$$

on $\mathcal{C}(a)$ respectively on the Cartan space $\mathcal{C}(a)$ [25].

Thus, a non-degenerate Monge-Ampère equation generates two 2-dimensional distributions on J^1M, which are formed by eigenspaces of the operator A_{ω_a}.

These eigenspaces $\mathcal{C}_+(a)$ and $\mathcal{C}_-(a)$ correspond to the eigenvalues 1 and -1 or ι and $-\iota$ in hyperbolic or elliptic cases respectively. Here $\iota = \sqrt{-1}$.

The distributions $\mathcal{C}_+$ and $\mathcal{C}_-$ are called *characteristic*.

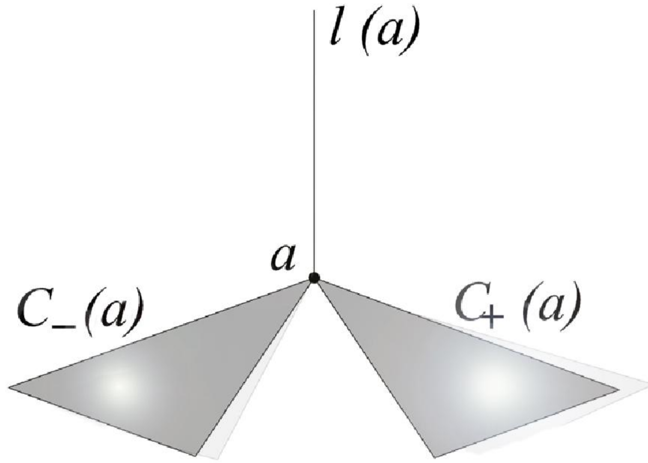

Figure 3.1: 3-tuple almost product structure for hyperbolic equations

The characteristic distributions are real for the hyperbolic equations and complex for the elliptic ones. For the elliptic equations they are also complex conjugate.

Moreover, planes $\mathcal{C}_+(a)$ and $\mathcal{C}_-(a)$ are skew-orthogonal with respect to the symplectic structure Ω_a.

On each of them the 2-form Ω_a is nondegenerate. It is easy to see that the first derivatives of the characteristic distributions $\mathcal{C}_\pm^{(1)} = \mathcal{C}_\pm + [\mathcal{C}_\pm, \mathcal{C}_\pm]$ are three-dimensional. Therefore their intersection $l = \mathcal{C}_+^{(1)} \cap \mathcal{C}_-^{(1)}$ is a one-dimensional distribution, which is transversal to $\mathcal{C}$ [25].

Hence, for the hyperbolic equations the tangent space $T_a(J^1M)$ splits into the direct sum

$$T_aJ^1M = \mathcal{C}_+(a) \oplus l(a) \oplus \mathcal{C}_-(a). \tag{3.7}$$

at each point $a \in J^1M$ (see Fig. 3.1) [25].

For elliptic equations we have the similar decomposition of the complexification of T_aJ^1M. But the distributions l is real also in this case.

We say that a non-degenerate equation is called *regular* if the derivatives $\mathcal{C}_\pm^{(k)}$ $(k = 1, 2, 3)$ of the characteristic distributions are distributions too.

The above decomposition of the tangent bundle allows us to construct a decomposition of the de Rham complex, which is generated by an equation [10, 25].

Denote the distributions $\mathcal{C}_+, l$, and $\mathcal{C}_-$ by P_1, P_2, and P_3, respectively.

Let $D(J^1M)$ be the module of vector fields on J^1M, and let D_j be the module of vector fields from the distribution P_j. Define the following submodules of $\Omega^s(J^1M)$:

$$\Omega_i^s = \{\alpha \in \Omega^s(J^1M) \mid\ X\rfloor\alpha = 0\ \forall\, X \in D_j,\ j \neq i\} \quad (i = 1, 2, 3).$$

We get the following decomposition of the module of differential s-forms on J^1M:

$$\Omega^s(J^1M) = \bigoplus_{|\mathbf{k}|=s} \Omega^{\mathbf{k}}, \tag{3.8}$$

where $\mathbf{k} =(k_1, k_2, k_3)$ is a multi-index, $k_i \in \{0, 1, \ldots, \dim P_i\}$, $|\mathbf{k}| = k_1+k_2+k_3$,

$$\Omega^{\mathbf{k}} = \left\{ \sum_{j_1+j_2+j_3=|\mathbf{k}|} \alpha_{j_1} \wedge \alpha_{j_2} \wedge \alpha_{j_3},\ \text{where } \alpha_{j_i} \in \Omega_i^{k_i} \right\} \subset \bigotimes_{i=1}^{3} \Omega_i^{k_i}.$$

Three first terms of this decomposition are shown on the diagram (see Fig. 3.2).

The exterior differential also splits into the direct sum of operators

$$d = \bigoplus_{|\mathbf{t}|=1} d_{\mathbf{t}},$$

where

$$d_{\mathbf{t}} : \Omega^{\mathbf{k}} \to \Omega^{\mathbf{k}+\mathbf{t}}.$$

Theorem 3.1. *1. The operators $d_{\mathbf{t}}$ are differentiations, i.e. the Leibnitz rule holds:*

$$d_{\mathbf{t}}(\alpha \wedge \beta) = d_{\mathbf{t}}\alpha \wedge \beta + (-1)^{\deg \alpha}\, \alpha \wedge d_{\mathbf{t}}\beta, \tag{3.9}$$

where $\alpha \in \Omega^{\mathbf{k}}$ and $\beta \in \Omega^{\mathbf{i}}$.

2. If the sum of negative components of the multi-index $\mathbf{t}$ less than -1 then $d_{\mathbf{t}} = 0$.

3. If the multi-index $\mathbf{t}$ contains one negative component and this component is -1 then operator $d_{\mathbf{t}}$ is a $C^\infty(N)$-homomorphism, i.e.,

$$d_{\mathbf{t}}(f\alpha) = f d_{\mathbf{t}}\alpha \tag{3.10}$$

for any dunction f and any differential form $\alpha \in \Omega^{\mathbf{k}}$.

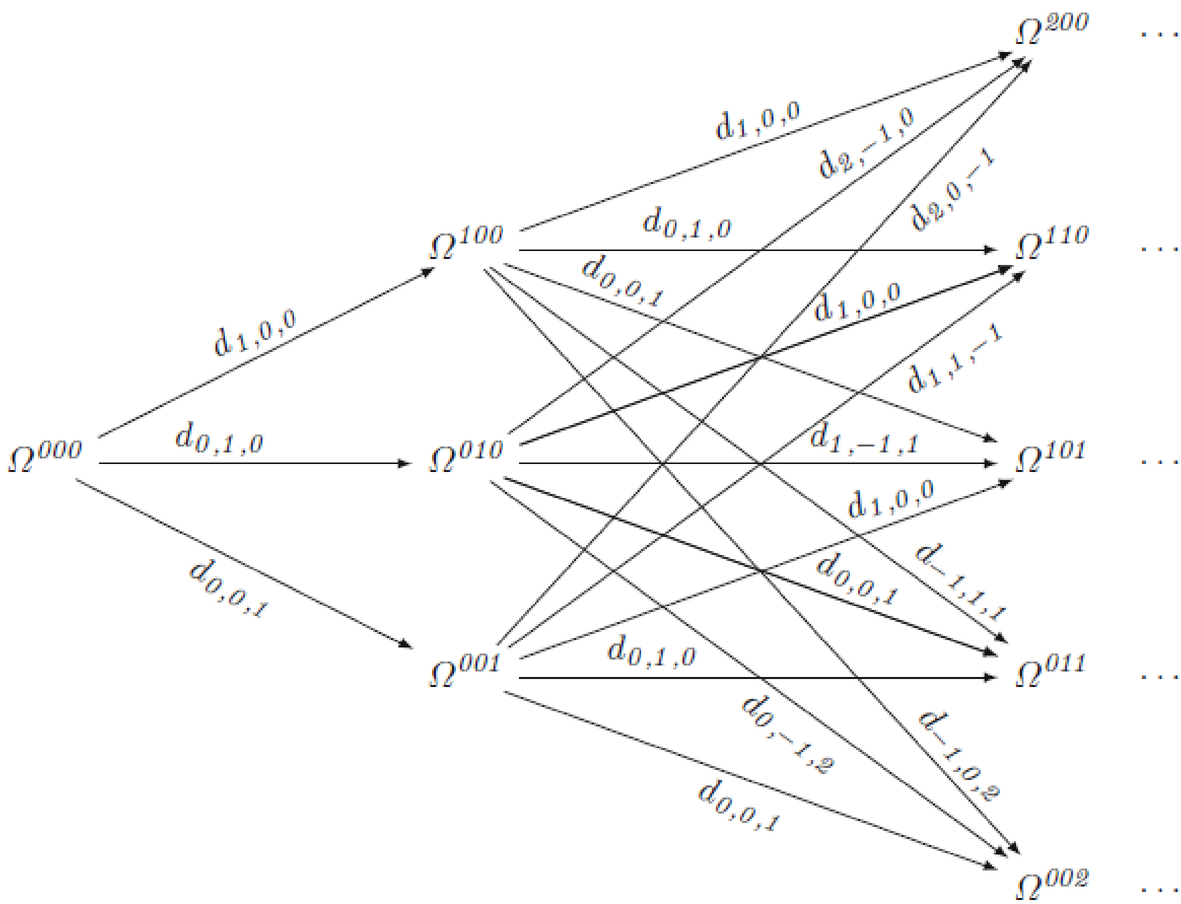

Figure 3.2: Decomposition of the de Rham complex on J^1M

Due to this theorem, we have the following eight homomorphisms:

$$d_{-1,1,1}, \quad d_{1,-1,1}, \quad d_{1,1,-1}, \quad d_{0,-1,2}, \quad d_{2,-1,0}, \quad d_{-1,0,2}, \quad d_{2,0,-1} \quad d_{2,0,-1}.$$

and four of them are zero. The nontrivial homomorphisms are the following:

$$d_{-1,1,1}, d_{1,1,-1}, d_{2,0-1}, \text{ and } d_{-1,0,2}.$$

Consider a case

$$\mathbf{t} = \mathbf{1}_j + \mathbf{1}_k - \mathbf{1}_s.$$

Then the differential $d_{\mathbf{t}}$ is a $C^\infty(N)$-homomorphisn. Due to Leibnitz's rule, this differential is completely defined by its values on $\Omega^1(N)$.

Note that

$$d_{\mathbf{1}_j+\mathbf{1}_k-\mathbf{1}_s} : \Omega^{\mathbf{1}_q} \to 0,$$

if $q \neq s$. Therefore a non-trivial $d_{\mathbf{1}_j+\mathbf{1}_k-\mathbf{1}_s}$ is a restriction to the module $\Omega^{\mathbf{1}_s}$:

$$d_{\mathbf{1}_j+\mathbf{1}_k-\mathbf{1}_s} : \Omega^{\mathbf{1}_s} \to \Omega^{\mathbf{1}_j} \wedge \Omega^{\mathbf{1}_k}.$$

In other words, these homomorphisms define tensor fields of a type (2,1) on N. We denote them by $\tau_{\mathbf{1}_j+\mathbf{1}_k-\mathbf{1}_s}$:

$$\tau_{\mathbf{1}_j+\mathbf{1}_k-\mathbf{1}_s} \in \Omega^{\mathbf{1}_j} \wedge \Omega^{\mathbf{1}_k} \otimes D_s.$$

Note that

$$\tau_{\mathbf{1}_j+\mathbf{1}_k-\mathbf{1}_s} : \Omega^{\mathbf{1}_s} \to \Omega^{\mathbf{1}_j} \wedge \Omega^{\mathbf{1}_k}$$

coinsides with operator $d_{\mathbf{1}_j+\mathbf{1}_k-\mathbf{1}_s}$.

Tensor fields $\tau_{\mathbf{1}_j+\mathbf{1}_k-\mathbf{1}_s}$ are differential invariants of Monge-Ampère equations.

So, we have four tensors of (2,1)–type:

$$\tau_{2,-1,0}, \quad \tau_{0,-1,2}, \quad \tau_{-1,1,1} \quad \text{and} \quad \tau_{1,1,-1}. \tag{3.11}$$

Example 15. *Coordinate representation of the tensor invariants for the hyperbolic Monge-Ampère equations*

$$v_{xy} = f\left(x, y, v, v_x, v_y\right), \tag{3.12}$$

has the following form:

$$
\begin{aligned}
\tau_{-1,1,1} = &(ff_{p_2p_2}dq_1 \wedge du - f_{p_2p_2}dp_2 \wedge du - p_1 f_{p_2p_2}dq_1 \wedge dp_2 - p_2 f_{p_2p_2}dq_2 \wedge dp_2 + \\
&(f_u - p_2 f_{p_2u} + f_{p_1}f_{p_2} - ff_{p_1p_2} - f_{q_2p_2})\, dq_2 \wedge du + \\
&(p_1 f_u - p_1p_2 f_{p_2u} - p_2 ff_{p_2p_2} + p_1 f_{p_1}f_{p_2} - p_1 ff_{p_1p_2} - p_1 f_{q_2p_2})\, dq_1 \wedge dq_2) \\
&\otimes \frac{\partial}{\partial p_1}, \\
\tau_{1,1,-1} = &(ff_{p_1p_1}dq_2 \wedge du - f_{p_1p_1}dp_1 \wedge du - p_1 f_{p_1p_1}dq_1 \wedge dp_1 - p_2 f_{p_1p_1}dq_2 \wedge dp_1 + \\
&(f_u + f_{p_1}f_{p_2} - p_1 f_{p_1u} - ff_{p_1p_2} - f_{q_1p_1})\, dq_1 \wedge du + \\
&(-p_2 f_u - p_2 f_{p_1}f_{p_2} + p_1p_2 f_{p_1u} + p_2 ff_{p_1p_2} + p_1 ff_{p_1p_1} + p_2 f_{q_1p_1})\, dq_1 \wedge dq_2) \\
&\otimes \frac{\partial}{\partial p_2}, \\
\tau_{2,-1,0} = &(dq_1 \wedge dp_1 - f_{p_2}dq_1 \wedge du + (p_2 f_{p_2} - f)\, dq_1 \wedge dq_2) \\
&\otimes \left(\frac{\partial}{\partial u} + f_{p_2}\frac{\partial}{\partial p_1} + f_{p_1}\frac{\partial}{\partial p_2}\right), \\
\tau_{0,-1,2} = &(dq_2 \wedge dp_2 - f_{p_1}dq_2 \wedge du - (p_1 f_{p_1} - f)\, dq_1 \wedge dq_2) \\
&\otimes \left(\frac{\partial}{\partial u} + f_{p_2}\frac{\partial}{\partial p_1} + f_{p_1}\frac{\partial}{\partial p_2}\right).
\end{aligned}
$$

3.3 The Laplace forms

Let's define a bracket $\langle \cdot, \cdot \rangle$ by the formula

$$\langle \alpha \otimes X, \beta \otimes Y \rangle = (Y \rfloor \alpha) \wedge (X \rfloor \beta)$$

for tensors $\alpha \otimes X$ and $\beta \otimes Y$.

Then differential 2-forms

$$\lambda_+ = \langle \tau_{0,-1,2}, \tau_{1,1,-1} \rangle, \qquad \lambda_- = \langle \tau_{2,-1,0}, \tau_{-1,1,1} \rangle. \tag{3.13}$$

we call *Laplace forms* or *Laplace invariants* of the Monge-Ampère equations E_ω.

Remark 2. *For the elliptic equations the Laplace forms are complex conjugate.*

Example 16. *For equation (3.12), the Laplace forms have the following coordinate representation:*

$$\lambda_- = f_{p_2p_2}\left(f_{p_1}dq_1 \wedge du - dq_1 \wedge dp_2\right) + \\ \left(f_u - p_2 f_{p_2u} + f_{p_1}f_{p_2} - p_2 f_{p_1} f_{p_2p_2} - f f_{p_1p_2} - f_{q_2p_2}\right) dq_1 \wedge dq_2, \tag{3.14}$$

$$\lambda_+ = f_{p_1p_1}\left(f_{p_2}dq_2 \wedge du - dq_2 \wedge dp_1\right) + \\ \left(-f_u + p_1 f_{p_1u} - f_{p_1}f_{p_2} + p_1 f_{p_2} f_{p_1p_1} + f f_{p_1p_2} + f_{q_1p_1}\right) dq_1 \wedge dq_2. \tag{3.15}$$

In particular, for linear equation

$$v_{xy} = a(x,y)v_x + b(x,y)v_y + c(x,y)v + g(x,y), \tag{3.16}$$

the Laplace forms are

$$\lambda_- = k dx \wedge dy \quad \text{and} \quad \lambda_+ = -h dx \wedge dy, \tag{3.17}$$

where

$$k = ab + c - b_y \qquad\qquad h = ab + c - a_x \tag{3.18}$$

are the classical Laplace invariants. This observation justifies our definition.

Remark that the classical Laplace invariants (3.18) of equation (3.16) are not absolute invariants in contrast to the forms $\lambda_\pm$.

Example 17. *For the linear elliptic equation*

$$v_{xx} + v_{yy} = a(x,y)v_x + b(x,y)v_y + c(x,y)v + g(x,y) \tag{3.19}$$

the Laplace forms are

$$\lambda_\pm = \frac{1}{4}\left(b_x - a_y \pm \left(\frac{1}{2}(a^2+b^2) + 2c - a_x - b_y\right)\iota\right) dx \wedge dy. \tag{3.20}$$

The coefficients

$$K = b_x - a_y, \qquad \text{and} \qquad H = \frac{1}{2}(a^2+b^2) + 2c - a_x - b_y \tag{3.21}$$

of these forms are the Cotton invariants [1].

3.4 Contact linearization of the Monge-Ampere equations

Consider the following problem:

Find a class of the Monge-Ampère equations that are locally contact equivalent to the linear equations

$$v_{xx} \pm v_{yy} = a(x,y)v_x + b(x,y)v_y + c(x,y)v + g(x,y). \tag{3.22}$$

A solution of the problem can be conveniently formulated in terms of the Laplace forms.

We consider three possible cases.

3.4.1 $\lambda_+ = \lambda_- = 0$

It is well known that if the classical Lagrange invariants h and k of a linear hyperbolic equation are zero, then the equation can be reduced to the wave equation (see [**?**], for example).

Similar statement is true for the Monge-Ampère equations.

Theorem 3.2 ([13]). *A hyperbolic Monge-Ampère equation is locally contact equivalent to the wave equation $v_{xy} = 0$ if and only if its Laplace invariants are zero: $\lambda_+ = \lambda_- = 0$.*

Corollary 2. *The equation*

$$v_{xy} = f\left(x, y, v, v_x, v_y\right)$$

is locally contact equivalent to the wave equation $v_{xy} = 0$ if and only if the function f has the following form:

$$f = \varphi_y v_x + \varphi_x v_y + (\varphi_v + \Phi_v) v_x v_y + R,$$

where the function $R = R(x, y, v)$ satisfies to the following ordinary linear differential equation:

$$R_v = (\varphi_v + \Phi_v)R + \varphi_{xy} - \varphi_x \varphi_y.$$

This equation can be solved:

$$R = e^{\varphi+\Phi} \left(\int (\varphi_{xy} - \varphi_x\varphi_y) e^{-\varphi-\Phi} dv + g \right).$$

Here $\varphi = \varphi(x, y, v)$, $\Phi = \Phi(v)$, and $g = g(x, y)$ are arbitrary functions.

Theorem 3.3 ([13]). *An elliptic Monge-Ampère equation is locally contact equivalent to the Poisson equation* $v_{xx} + v_{yy} = f(x, y)$ *if and only if its Laplace invariants are zero:* $\lambda_+ = \lambda_- = 0$.

If, in addition, coefficients of the Monge-Ampère equation are analytic functions, then the equation is locally contact equivalent to the Laplace equation $v_{xx} + v_{yy} = 0$.

3.4.2 $\lambda_+ \neq 0$ and $\lambda_- \neq 0$

Note that for the Laplace invariants of the linear equations (see (3.17) and (3.20)) the following conditions:

$$\lambda_+ \wedge \lambda_+ = 0, \quad \lambda_- \wedge \lambda_- = 0, \quad \lambda_+ \wedge \lambda_- = 0, \quad \text{and} \quad d\lambda_+ = d\lambda_- = 0 \tag{3.23}$$

hold.

Hence, for the Monge-Ampère equations that are locally contact equivalent to equation (3.22) this is also true.

It follows from the following Theorem that conditions (3.23) are sufficient.

Theorem 3.4 ([10, 13]). *Suppose* $\lambda_+ \neq 0$ *and* $\lambda_- \neq 0$. *A nondegenerate Monge-Ampère equation is locally contact equivalent to equation (3.22) if and only if conditions (3.23) hold.*

3.4.3 One of the Laplace forms is zero and the another one is not

Due to Remark 2, this case realizes only for the hyperbolic equations.

For definiteness, suppose that $\lambda_- = 0$ and $\lambda_+ \neq 0$. We shall suppose $\lambda_+ \wedge \lambda_+ = 0$ because this condition holds for the linear equations. This means that $\lambda_+ = \eta_- \wedge \vartheta_+$, where $\eta_- \in \Omega^{001}$ and $\vartheta_+ \in \Omega^{100}$ are differential 1-forms.

Theorem 3.5 (see [13]). *Suppose that one of the Laplace forms is zero and the another one, say* λ_+, *is not. A hyperbolic Monge-Ampère equation is locally contact equivalent to a linear equation if and only if* $d\lambda_+ = 0$, $\lambda_+ = \eta_- \wedge \vartheta_+$, *and the distribution* $\mathcal{F}\langle\vartheta_+\rangle$ *is completely integrable.*

3.5 The Hunter-Saxton equation

Consider the Hunter-Saxton equation

$$v_{tx} = vv_{xx} + \kappa v_x^2, \tag{3.24}$$

where κ is a constant. This equation is hyperbolic, and it has applications in the theory of a director field of a liquid crystal [5].

The corresponding effective differential 2-form and the operator A_ω are the following:

$$\omega = 2udq_2 \wedge dp_1 + dq_1 \wedge dp_1 - dq_2 \wedge dp_2 - 2\kappa p_1^2 dq_1 \wedge dq_2$$

and

$$A_\omega = \begin{Vmatrix} 1 & 2u & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -2\kappa p_1^2 & 1 & 0 \\ 2\kappa p_1^2 & 0 & 2u & -1 \end{Vmatrix}.$$

Let us choice the following bases in the module of vector fields on J^1M:

$$\begin{aligned}
X_1 &= \frac{\partial}{\partial q_1} + p_1\frac{\partial}{\partial u} + \kappa p_1^2 \frac{\partial}{\partial p_2}, \\
X_2 &= \frac{\partial}{\partial p_1} + u\frac{\partial}{\partial p_2}, \\
Z &= \frac{\partial}{\partial u} + (2\,\kappa - 1)\, p_1 \frac{\partial}{\partial p_2}, \\
Y_1 &= \frac{\partial}{\partial q_2} + \kappa p_1^2 \frac{\partial}{\partial p_1} - u\frac{\partial}{\partial q_1} + (p_2 - up_1)\frac{\partial}{\partial u}, \\
Y_2 &= \frac{\partial}{\partial p_2}.
\end{aligned}$$

The dual basis of the module of differential 1-forms is

$$\begin{aligned}
\alpha_1 &= dq_1 + udq_2, \\
\alpha_2 &= dp_1 - \kappa p_1^2 dq_2, \\
\theta &= du - p_1 dq_1 - p_2 dq_2, \\
\beta_1 &= dq_2, \\
\beta_2 &= dp_2 + (1 - 2\kappa)\, p_1 du + (\kappa - 1)\, p_1^2 dq_1 + (2\kappa - 1)\, p_1 p_2 dq_2 - udp_1.
\end{aligned}$$

The vector fields X_1, X_2 and Y_1, Y_2 form bases of the modles $D(\mathcal{C}_+)$ and $D(\mathcal{C}_-)$ respectively.

Tensor invariants of equation (3.24) have the form:

$$\tau_{-1,1,1} = -\left(p_1 dq_1 \wedge dq_2 + dq_2 \wedge du\right) \otimes \left(\frac{\partial}{\partial q_1} + p_1 \frac{\partial}{\partial u} + \kappa p_1^2 \frac{\partial}{\partial p_2}\right),$$

$$\tau_{1,1,-1} = 2(\kappa - 1)\,\big(\kappa p_1^3 dq_1 \wedge dq_2 + \kappa p_1^2 dq_2 \wedge du -$$
$$dp_1 \wedge du - p_1 dq_1 \wedge dp_1 - p_2 dq_2 \wedge dp_1\big) \otimes \frac{\partial}{\partial p_2},$$

$$\tau_{2,-1,0} = \big(dq_1 \wedge dp_1 - \kappa p_1^2 dq_1 \wedge dq_2 + u dq_2 \wedge dp_1\big) \otimes$$
$$\left(\frac{\partial}{\partial u} + (2\kappa - 1)\, p_1 \frac{\partial}{\partial p_2}\right),$$

$$\tau_{0,-1,2} = \big(dq_2 \wedge dp_2 + (1 - 2\kappa)\, p_1 dq_2 \wedge du + (1 - \kappa)\, p_1^2 dq_1 \wedge dq_2 - u dq_2 \wedge dp_1\big) \otimes$$
$$\left(\frac{\partial}{\partial u} + (2\kappa - 1)\, p_1 \frac{\partial}{\partial p_2}\right).$$

Then the Laplace forms for the Hunter-Saxton equation are

$$\lambda_- = -dq_2 \wedge dp_1,$$
$$\lambda_+ = 2\,(1 - \kappa)\, dq_2 \wedge dp_1.$$

Due to Theorem 3.4, the equation is linearized. The required transformation has the form

$$Q_1 = \kappa q_2 + \frac{1}{p_1}, \quad Q_2 = q_2, \quad U = u - p_1 q_1, \quad P_1 = q_1 p_1^2, \quad P_2 = p_2 - \kappa q_1 p_1^2.$$

This transformation takes the effective form ω to the form

$$\Omega = dQ_1 \wedge dP_1 - dQ_2 \wedge dP_2 + \left(\frac{2(2\kappa - 1)P_1}{\kappa Q_2 - Q_1} + \frac{2U}{(\kappa Q_2 - Q_1)^2}\right) dQ_1 \wedge dQ_2.$$

The corresponding linear equation is the Euler-Poisson equation

$$U_{Q_1 Q_2} = \frac{2\kappa - 1}{Q_1 - \kappa Q_2} U_{Q_1} - \frac{1}{(Q_1 - \kappa Q_2)^2} U.$$

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