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Definability properties and the congruence closure

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Abstract. We introduce a natural class of quantifiers **Th** containing all monadic type quantifiers, all quantifiers for linear orders, quantifiers for isomorphism, Ramsey type quantifiers, and plenty more, showing that no sublogic of $L_{\omega\omega}(\mathbf{Th})$ or countably compact regular sublogic of $L_{\infty\omega}(\mathbf{Th})$, properly extending $L_{\omega\omega}$, satisfies the uniform reduction property for quotients. As a consequence, none of these logics satisfies either Δ-interpolation or Beth's definability theorem when closed under relativizations. We also show the failure of both properties for any sublogic of $L_{\infty\omega}(\mathbf{Th})$ in which Chang's quantifier or some cardinality quantifier Q_α, with $\alpha \geqq 1$, is definable.

Introduction

Under reasonable closure conditions any logic good for classifying classical first order structures of finite type is of the form $L = L_{\omega\omega}(Q_i \mid i \in I)$ where the Q_i are Lindström quantifiers. A logic is *regular* if it is of the above form, and has in addition the relativization property (cf. [E]).

In [C1] it is shown that no regular proper extension of $L_{\omega\omega}$ generated by quantifiers of monadic type satisfies Craig's interpolation lemma. On the other hand we have the result by Mekler and Shelah [MeSh1] to the effect that the validity of the weak Beth property for $L_{\omega\omega}(Q_1)$ is consistent with **ZF**. This raises the obvious question about the status of the intermediate definability properties: Δ-interpolation (or Souslin-Kleene property) and Beth's definability theorem for these logics. An earlier result of Friedman [Fr] shows that Beth property fails when L is any logic between $L_{\omega\omega}(Ch)$ or $L_{\omega\omega}(Q_\alpha)$, $\alpha \geqq 1$, and the infinitary logic

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$L_{\infty\omega}(Ch, Q_\alpha \mid \alpha \geqq 0)$, where Ch is Chang's quantifier and the Q_α are the cardinality quantifiers. Other negative results appear scattered in the literature.

In this paper we introduce a natural class of quantifiers **Th** which includes not only all monadic type quantifiers but also all linear order quantifiers, as Shelah's cofinality quantifiers or quantifiers comparing linear orders [Sh1], isomorphism quantifiers (see survey in [Mu]) and many others one could fancy, say:

$$Sxy\varphi(x,y) \Leftrightarrow \varphi(x,y) \text{ defines a Souslin tree,}$$

$$Gxyz\varphi(x,y,z) \Leftrightarrow \varphi(x,y,z) \text{ defines the operation of a finitely generated group,}$$

$$R^n_\kappa x_1, \dots, x_n \varphi(x_1, \dots, x_n) \Leftrightarrow \text{there is a set of } \kappa \text{ indiscernibles for } \varphi \text{ in the field of } \varphi,$$

and we show, via a uniform counterexample, that both properties fail for any regular sublogic of $L_{\omega\omega}(\mathbf{Th})$, or for any countably compact regular sublogic of $L_{\infty\omega}(\mathbf{Th})$ properly extending $L_{\omega\omega}$. They also fail for any L (not necessarily regular) between $L_{\omega\omega}(Ch)$ or $L_{\omega\omega}(Q_a)$, $a \geqq 1$, and $L_{\infty\omega}(\mathbf{Th})$.

The canonical reason for failure of Δ-interpolation is that these logics do not have the *uniform reduction* property with respect to the *quotient* operation of dividing a structure by a congruence relation [Fe, Ma]. In other words, these logics are not *congruence closed* in the sense of [MeSh2]. The counterexample for Δ-interpolation is transformed in one for Beth property using a tree technique due to Friedman [Fr], and later generalized in [MaSh1, 2].

Recently Hella [H] has shown strong results which intersect at some points with ours, implying for example that $\Delta L_{\omega\omega}(Q_\alpha, Q_{\alpha+1})$ and Beth $L_{\omega\omega}(Q_\alpha)$ are not finitely generated for regular ω_α.

All logics considered in this paper will be single sorted and will have finite occurrence number. For all unexplained concepts and notation we refer the reader to [BFe].

1 Thin quantifiers

A set I is said to be a *set of indiscernibles* for a n-ary relation R if $I \subseteq \text{Field}(R)$, and for any $x_1, \dots, x_n$, $y_1, \dots, y_n \in I$ with $x_i \neq x_j$, $y_i \neq y_j$ for $i \neq j$ we have: $(x_1, \dots, x_n) \in R \Leftrightarrow (y_1, \dots, y_n) \in R$. Here, $\text{Field}(R) = \bigcup_i \pi_i(R)$, π_i being the i^{th} projection, $i = 1, \dots, n$.

A Lindström quantifier Q will be said to be κ-*thin* if $(A, R_1, \dots, R_m) \in Q$ with $|A| \geqq \kappa$ implies none of the R_i, $i = 1, \dots, m$, admits a set of $|A|$ indiscernibles (in its field). Let $\mathbf{T}_\kappa$ be the class of all κ-thin quantifiers. Evidently $\mathbf{T}_\kappa \subset \mathbf{T}_{\kappa'}$, for $\kappa < \kappa'$.

Recall that for a class of quantifiers $\mathbf{C}$, the fragment of $L_{\infty\omega}(\mathbf{C})$ consisting of formulae of quantifier rank less than an ordinal α is denoted by $L^\alpha_{\infty\omega}(\mathbf{C})$, where a quantifier of arity $\langle n_1, \dots, n_s \rangle$ is supposed to increase the quantifier rank by $\max\{n_i\}$.

A Lindström quantifier Q will be called *thin* if there is $n \in \omega$ and a cardinal κ_0 such that for each $\kappa \geqq \kappa_0$, Q is definable in models of power κ by a sentence of $L^n_{\infty\omega}(\mathbf{T}_\kappa)$, depending only on κ.

Let **Th** be the class of all thin quantifiers. It contains $\bigcup_\kappa \mathbf{T}_\kappa$ and it is closed under all first order operations. Many familiar quantifier belong to **Th**.

Examples

(a) *Cardinality quantifiers.* $\neg Q_\alpha$ is ω_α-thin, because a set of $|A|$ many indiscernibles for a monadic relation R in A is just a subset of Field$(R)=R$, of power $|A|$, which is impossible if $(A, R)\in \neg Q_\alpha$ and $|A|\geqq \omega_\alpha$. Similarly, $Q^{=\kappa}$ (there are exactly κ...) is κ^+-thin, and $\neg Ch$ is 1-thin.

(b) *Linear order quantifiers.* Those quantifiers Q for which $(A, R_1, \ldots, R_n)\in Q$ implies the R_i are linear orders of their fields are 2-thin, due to antisymmetry and tricotomy. Various of these quantifiers have been discussed in the literature, for example $Wxy\varphi(x,y)$, $Q_\kappa^{\mathrm{cof}}\varphi(x,y)$, or $Ixyzx(\varphi(x,y), \psi(z,2))$, meaning respectively: φ defines a well order, φ defines a linear order of cofinality κ, or φ and ψ define isomorphic linear orders. We still obtain thin quantifiers allowing the R_i to be *partial orders of bounded width*, for example:

$$S_\kappa xy\varphi(x,y)\Leftrightarrow \varphi \text{ defines a } \kappa\text{-Souslin tree in its field}$$

is κ-thin because of the κ-chain condition.

(c) *Bounded power quantifiers.* Q is bounded in power if there is κ such that for all $(A, R_1, \ldots, R_n)\in Q$, $|R_i|<\kappa$, $i=1,\ldots,n$. Such quantifiers are obviously κ-thin. They include the quantifiers $\neg Q_\alpha$ and $Q^{=\kappa}$; also the *quantifiers for isomorphism* to a given structure $Q^{I\mathfrak{B}}$:

$$\mathfrak{A}\models Q^{I\mathfrak{B}}x\bar{x}_1\ldots\bar{x}_m(\varphi,\varphi_1,\ldots,\varphi_m)\Leftrightarrow(\varphi^{\mathfrak{A}},\varphi_1^{\mathfrak{A}},\ldots,\varphi_m^{\mathfrak{A}})\approx\mathfrak{B},$$

and many others as:

$$Gxyz\varphi(x,y,z)\Leftrightarrow\varphi \text{ defines the operation of a finitely generated group.}$$

(d) *Ramsey quantifiers.* The quantifiers $R_\kappa^n x_1\ldots x_n\varphi(x_1,\ldots,x_n)$ meaning there is a set of κ indiscernibles for φ (in its field). Obviously, $\neg R_\kappa^n$ is κ-thin.

Lemma 1. Th *contains all monadic type quantifiers.*

Proof. A monadic type quantifier Q is characterized by a family of sets $c(\kappa)\subseteq P(\kappa)^{2^n}$, $\kappa\in$ Card., such that $(A, P_1, \ldots, P_n)\in Q$ if and only if $(|R_1|, \ldots, |R_{2^n}|)\in c(|A|)$, where the R_j are the blocks of the partition induced by the P_i in A. Hence, in models of power κ:

$$Qx_1\ldots x_n(P_1(x_1),\ldots,P_n(x_n))\Leftrightarrow\bigvee_{\beta\in c(\kappa)}\bigwedge_{j=1}^{2^n}Q^{=\beta_j}=xR_j(x),$$

which belongs in said structures to $L^1_{\infty\omega}(\mathrm{T}_\kappa)$ because $Q^{=\beta_j}$ is κ-thin for $\beta_j<\kappa$ and is equivalent to Q_α for $\beta_j=\kappa=\omega_\alpha$. $\square$

2 Failure of uniform reduction for quotients

Recall that for any logic L there are minimal closures ΔL and Beth L satisfying Δ-interpolation and Beth definability property, respectively. In general both closures are incomparable (cf. [E]).

Other sorts of closures may be defined. A logic is said to be *congruence closed* [MeSh2] if for any sentence φ in L of type say τ there is another sentence φ^{-q} in L of type $\tau\cup\langle E\rangle$, E a new binary symbol, such that:

$$\mathfrak{A}/\bar{E}\models\varphi\Leftrightarrow(\mathfrak{A},\bar{E})\models\varphi^{-q} \tag{1}$$

for any τ-structure $\mathfrak{A}$ and congruence relation $\bar{E}$ in $\mathfrak{A}$. It is easy to see that this is the same as asking the logic to have the *uniform reduction property* with respect to the quotient operation:

$$F(\mathfrak{A}, \bar{E}) = \begin{cases} \mathfrak{A}/\bar{E} & \text{if } \bar{E} \text{ is a congruence in } \mathfrak{A} \\ \mathfrak{A} & \text{otherwise,} \end{cases}$$

in the sense of [Fe, Ma]. For any logic L there is a well behaved congruence closure qL, obtained by adjoining to L the sentences defined by (1) as new quantifiers.

In the next theorem no regularity condition is asked from the logic L, it could consist of a single sentence added to $L_{\omega\omega}$.

Theorem 1. *If L extends properly $L_{\omega\omega}$ then $qL \nleq L_{\omega\omega}(\mathbf{Th})$. If $qL \leqq L_{\infty\omega}(\mathbf{Th})$ then L has the Karp property.*

Proof. Given any cardinal κ and structure $\mathfrak{A}$ of type τ, define $\mathfrak{A}(\kappa) = (\mathfrak{A} \times \kappa_\tau, E)$ where $\mathfrak{A} \times \kappa_\tau$ is the ordinary cartesian product of structures, κ_τ is the structure with universe κ and relations $R^{\kappa_\tau} = \kappa^{n_R}$ for each $R \in \tau$, and E is the relation: $(a, \alpha)E(b, \beta) \Leftrightarrow a = b$. Obviously, E is a congruence in $\mathfrak{A}(\kappa)$ such that $\mathfrak{A}(\kappa)/E \approx (\mathfrak{A}, =)$, and if $E(x)$ denotes the E-equivalence class of $x \in A \times \kappa$ then $|E(x)| = \kappa$.

Claim 1. Let $\varphi(v_1, \ldots, v_{r+n})$ be a formula in any logic, $\bar{a} = \{a_1, \ldots, a_n\} \subseteqq A \times \kappa$, and

$$\Phi = \{\bar{x} \in (A \times \kappa)^r \mid \mathfrak{A}(\kappa) \models \varphi(\bar{x}, a_1, \ldots, a_n)\},$$

then one of the two following properties must hold:

i) $\mathrm{Field}(\Phi) \subseteqq \bar{a}$.

ii) There is $x \in A \times \kappa$ such that $E(x) \setminus \bar{a}$ is a set of indiscernibles for Φ.

Proof. Since E is a congruence, for any $x \in A \times \kappa$ and $x_1, \ldots, x_r, y_1, \ldots, y_r \in E(x) \setminus \bar{a}$ with $x_1 \rightsquigarrow y_i$ bijective, there is an automorphism of $\mathfrak{A}(\kappa)$ sending x_i to y_i and leaving fixed $\bar{a}$. If there exists $x \in \mathrm{Field}(\Phi) \setminus \bar{a}$, then by the isomorphism axiom of logics applied to the formula defining $\mathrm{Field}(\Phi)$ we have $E(x) \setminus \bar{a} \subseteqq \mathrm{Field}(\Phi)$. By the same axiom applied to φ we conclude this is a set of indiscernibles for $\mathrm{Field}(\Phi)$.

Claim 2. For $\kappa \geqq \omega$, any formula of $L^\alpha_{\infty\omega}(\mathbf{T}_\kappa)$ is equivalent in all structures $\mathfrak{A}(\kappa)$ to a formula of $L^\alpha_{\infty\omega}$ depending only on $|A \times \kappa|$.

Proof. If $\mathfrak{A}(\kappa) \models Qx_1 \ldots x_s(\varphi_1(\bar{x}_1, \bar{a}), \ldots, \varphi_s(\bar{x}_s, \bar{a}))$ for a κ-thin quantifier Q, then by Claim 1, $\mathrm{Field}(\Phi_i) \subseteqq \bar{a}$ for any of the truth sets Φ_i corresponding to the φ_i. If Q has arity $\langle n_1, \ldots, n_s \rangle$ let $\kappa' = |A \times \kappa|$, and

$$Q(\kappa') = \{(R_1, \ldots, R_s) \mid (\kappa', R_1, \ldots, R_s) \in Q, R_i \subseteqq \{0, \ldots, n\}^{n_i}\},$$

then by the closure of quantifiers under isomorphism plus the fact that $\mathfrak{A}(\kappa)$ has power κ', the above formula is equivalent in this structure to the disjunction over $Q(\kappa')$ of

$$\bigwedge_{i=1}^{s} \forall \bar{x}_i \left[\varphi_i(\bar{x}_i, \bar{a}) \leftrightarrow \bigvee_{j \in R_i} \bigwedge_{r=1}^{n_i} (x_i^r = a_{j_r}) \right],$$

where $\bar{x}_i = (x_i^1, \ldots, x_i^{n_i})$. This formula adds $\max\{n_i\}$ to the rank of the φ_i, which is exactly what the quantifier Q does.

Now we may prove the theorem. Let $\varphi \in L \setminus L_{\omega\omega}$ be of finite type τ, then by a well known argument due to Lindström there are for each $m \in \omega$ structures $\mathfrak{A}_1$ and $\mathfrak{A}_2$

of type τ such that

$$\mathfrak{A}_1 \equiv \mathfrak{A}_2 \quad \text{in} \quad L^m_{\infty\omega}, \qquad \mathfrak{A}_1 \models \varphi, \qquad \mathfrak{A}_2 \not\models \varphi. \tag{2}$$

By using partial isomorphisms one may easily show that for any κ

$$\mathfrak{A}_1(\kappa) \equiv \mathfrak{A}_2(\kappa) \quad \text{in} \quad L^m_{\infty\omega}. \tag{3}$$

By Claim 2, one may conclude that if $\kappa \geqq |A_1|, |A_2|, \omega$,

$$\mathfrak{A}_1(\kappa) \equiv \mathfrak{A}_2(\kappa) \quad \text{in} \quad L^m_{\infty\omega}(\mathbf{T}_\kappa). \tag{4}$$

Now, any sentence $\sigma \in L_{\omega\omega}(\mathbf{Th})$ contains finitely many thin quantifiers, and so there is $m \in \omega$ and κ_0 such that for all $\kappa \geqq \kappa_0$, σ is equivalent in models of power κ to a sentence in $L^m_{\infty\omega}(\mathbf{T}_\kappa)$. Taking $\kappa \geqq |A_1|, |A_2|, \kappa_0, \omega$, we see by (4) that σ can not reduce uniformly φ, since $\mathfrak{A}_1(\kappa)/E_1 \models \varphi$ and $\mathfrak{A}_2(\kappa)/E_2 \not\models \varphi$. Hence $\varphi^{-q} \notin L_{\omega\omega}(\mathbf{Th})$.

Finally, if L is not a Karp logic, one may strengthen (2) to elementary equivalence of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ in $L_{\infty\omega}$, yielding equivalence with respect to $L_{\infty\omega}$ in (3), and with respect to $L_{\infty\omega}(\mathbf{T}_\kappa)$ in (4) by Claim 2. A sentence of $L_{\infty\omega}(\mathbf{Th})$ contains a *set* of thin quantifiers and so there is κ_0 such that for models of power $\kappa \geqq \kappa_0$ it is equivalent to a sentence in $L_{\infty\omega}(\mathbf{T}_\kappa)$. This, together with the strengthened version of (4) implies that $\varphi^{-q} \notin L_{\infty\omega}(\mathbf{Th})$. □

Corollary 1. *There is no congruence closed logic L with $L_{\omega\omega} < L \leqq L_{\omega\omega}(\mathbf{Th})$, neither countably compact, regular, congruence closed L with $L_{\omega\omega} < L \leqq L_{\infty\omega}(\mathbf{Th})$.*

Proof. The only countably compact, regular extension of $L_{\omega\omega}$ with the Karp property is $L_{\omega\omega}$ itself, by Lindström's argument coding partial isomorphisms. □

Compare this result with Theorem 4.8 in Krinicki [K]. There are many interesting congruence closed logics, which by the first part of the Corollary can not be sublogics of $L_{\omega\omega}(\mathbf{Th})$. For example $L_{\kappa\omega}$ with$\kappa \geqq \omega_1$ (these are sublogics of $L_{\infty\omega}(\mathbf{Th})$); $qL_{\omega\omega}(Q_\alpha) = L_{\omega\omega}(E_\alpha)$, where $E_\alpha xy\varphi(x, y)$ says that φ defines an equivalence relation with at least ω_α equivalence classes; $L_{\omega\omega}(Q^n_\alpha)$, $n \geqq 2$, where Q^n_α is the n-ary Magidor-Malitz quantifier in the ω_α-interpretation; monadic second order logics L^{m2}, L^{m2}_α, and $L^{m2}_{<\alpha}$, where the second order variables range over arbitrary monadic predicates, monadic predicates of power at least ω_α, and monadic predicates of power less than ω_α, respectively. See [K] for more examples and some proofs.

Väänänen shows in [V] that there is no family Q^i, $i \in \omega$, of monadic type quantifiers such that $L^{m2}_{<\alpha} \leq L_{\omega\omega}(Q^i \mid i \in \omega)$. This is explained and improved by our result. We can do better.

Corollary 2. *None of the following logics is a sublogic of $L_{\infty\omega}(\mathbf{Th})$: $L_{\omega\omega}(E_\alpha)$, $L_{\omega\omega}(Q^2_\alpha)$, L^{m2}_α, and $L^{m2}_{<\alpha}$, when $\alpha \geqq 1$, also L^{m2} and L^{m2}_0.*

Proof. These logics do not have the Karp property because either the quantifier Q_α, $\alpha \geqq 1$, or the class of complete dense linear orders is definable in them. □

3 Failure of Δ-interpolation

The logics introduced in [C2] are such that $\Delta L = L \leqq L_{\omega\omega}(\mathbf{Th})$. Hence, by Theorem 1 we have for them $qL \nleq \Delta L$. This is not possible if the logic has relativizations.

Lemma 2. *If L is an extension of $L_{\omega\omega}$ closed under conjunctions, negations, and relativizations then $qL \leqq \Delta L$.*

Proof. Let φ be a sentence of L of type τ and consider the sentences of $L_{\omega\omega}$ where E and V are new predicate symbols:

$$\theta\colon \forall x Exx \wedge \bigwedge_{R \in \tau \cup \{E\}} \forall \bar{x} \forall \bar{y} \Big(\bigwedge_i Ex_i y_i \to (R\bar{x} \to R\bar{y}) \Big)$$

$$\psi\colon \forall x \exists ! y (Vy \wedge Exy).$$

Then the following are complementary PC classes of type $\tau \cup \{E\}$ in L:

$$C_1 = \{\mathfrak{A} \mid \exists \bar{V} (\mathfrak{A}, \bar{V}) \models \theta \wedge \psi \wedge \varphi^V\},$$

$$C_2 = \{\mathfrak{A} \mid \exists \bar{V} (\mathfrak{A}, \bar{V}) \models \theta \to \psi \wedge \neg \varphi^V\},$$

and φ^{-q} is the sentence of ΔL making C_1 elementary, because $\mathfrak{A} \in C_1$ if and only if $E^{\mathfrak{A}}$ is a congruence in $\mathfrak{A}$, and there is a set of representatives $\bar{V}$ for $E^{\mathfrak{A}}$ such that $\mathfrak{A}/E^{\mathfrak{A}} \approx \mathfrak{A} \restriction \bar{V} \models \varphi$. As ΔL is always closed under substitutions the result follows. □

Theorem 2. *If L is a proper extension of $L_{\omega\omega}$ closed under relativizations then $\Delta L \nleqq L_{\omega\omega}(\mathbf{Th})$. If, in addition, ΔL is not a Karp logic, for example it is countably compact, then $\Delta L \nleqq L_{\infty\omega}(\mathbf{Th})$.*

Proof. It is easy to check that ΔL is regular, hence, by Lemma 2, $q\Delta L \leqq \Delta\Delta L = \Delta L$. The conclusion follows from Theorem 1. □

Many results in the literature become special cases of this theorem, as the following examples illustrate.

(a) $L_{\omega\omega}(Q^{I\mathfrak{A}})$ does not satisfy Δ-interpolation for any infinite structure $\mathfrak{A}$, moreover $\Delta L_{\omega\omega}(Q^{I\mathfrak{A}})$ can not be generated by thin quantifiers. Compare with Theorem 2.3.1 in Mundici [Mu].

(b) $\Delta L_{\omega\omega}(Q_\alpha) \nleqq L_{\infty\omega}(\mathbf{Th})$ for $\alpha \geqq 1$, compare with [KLV]. Also $\Delta L_{\omega_2\omega} \nleqq L_{\infty\omega}(\mathbf{Th})$ because Q_1 is definable in this logic.

(c) Let C be *any* non empty class of infinite cardinals and let $Q_C^{\mathrm{cof}} xy\varphi(x,y)$ mean that $\varphi(x,y)$ defines a linear order of cofinality in C, then $L_{\omega\omega}(Q_C^{\mathrm{cof}})$ does not satisfy Δ-interpolation. It is not a Karp logic since it allows elementary classes of uncountable dense linear order so $\Delta L_{\omega\omega}(Q_C^{\mathrm{cof}}) \nleqq L_{\infty\omega}(\mathbf{Th})$. Compare with [MaSh2]. We have similar results for the logic $L_{\omega\omega}(Q_C^{\mathrm{den}})$ where $Q_C^{\mathrm{den}} xy\varphi(x,y)$ means that $\varphi(x,y)$ defines a linear order having a dense subset of power belonging to C (cf. Theorem 3.2.1 in [Mu]).

(d) $L_{\omega\omega}(aa) \nleqq L_{\infty\omega}(\mathbf{Th})$ because $\Delta L_{\omega\omega}(Q_\omega^{\mathrm{cof}}) \leqq L_{\omega\omega}(aa)$ by one of the few known positive results on definability, shown by Shelah [Sh2]. □

The logic $L_{\omega\omega}(Ch)$ is not covered by Theorem 2 because it is not closed under relativizations. However, we can dispence with the relativization hypothesis if the logic is strong enough to define Ch or one of the Q_α, $\alpha \geqq 1$.

Theorem 3. *A sublogic of $L_{\infty\omega}(\mathbf{Th})$ extending $L_{\omega\omega} + Ch$ or $L_{\omega\omega} + Q_\alpha$ for some $\alpha \geqq 1$ does not satisfy Δ-interpolation.*

Proof. In case $Ch \in L$ consider the complementary PC classes C_1, C_2 defined as in the proof of Lemma 2, with $\tau = \emptyset$ and using $ChxXx$ instead of the relativized sentence φ^V. For any $\kappa \geqq \omega$, the structures $\mathfrak{A}_1 = (\kappa)$ and $\mathfrak{A}_2 = (\omega)$ are $L_{\infty\omega}$-

elementary equivalent, and so by Claim 2 in the proof of Theorem 1, $\mathfrak{A}_1(\kappa)\equiv\mathfrak{A}_2(\kappa)$ in $L_{\infty\omega}(\mathbf{T}_\kappa)$, but the first belongs to C_1 and the second to C_2; hence, these two classes are inseparable in $L_{\infty\omega}(\mathbf{Th})$. If $Q_\alpha\in L$ for some $\alpha\geqq 1$, use $Q_\alpha xV(x)$ instead of φ^V to define the classes, and let $\mathfrak{A}_1=(\omega_\alpha)$, $\mathfrak{A}_2=(\omega)$, then for any $\kappa\geqq\omega_\alpha$ $\mathfrak{A}_1(\kappa)\equiv\mathfrak{A}_2(\kappa)$ in $L_{\infty\omega}(\mathbf{T}_\kappa)$; but these two structures belong to distinct C_i. $\square$

4 Failure of Beth's property

Logics generated by thin quantifiers do not have in general the "tree preservation property", cf. [Ma], but preservation for certain structures will be enough to transform our counterexample to interpolation in one for Beth's property. This time we will assume regularity of the logic, although the reader may verify slightly weaker conditions suffice.

Theorem 4. *If L is a regular proper extension of $L_{\omega\omega}$ then Beth $L\nleqq L_{\omega\omega}(\mathbf{Th})$. If in addition L is not a Karp logic (for example it is countably compact) then Beth $L\nleqq L_{\infty\omega}(\mathbf{Th})$.*

Proof. Given $\varphi\in L\backslash L_{\omega\omega}$, let $C_i=\{\mathfrak{A}\mid\exists\bar{V}(\mathfrak{A},\bar{V})\models\sigma_i\}$, $i=1,2$, be the complementary PC classes of L introduced in the proof of Lemma 2, where σ_1 and σ_2 are sentences in the vocabulary $\tau\cup\{E,V\}$. Add a new binary relation symbol R and consider the following sentence $\Sigma(V)$ of type $\tau\cup\{E,R,V\}$, which belongs to L by regularity:

$$\forall x[(Vx\to\sigma_1^{\{y\mid Rxy\}})\wedge(\bar{V}x\to\sigma_2^{\{y\mid Rxy\}})].$$

Then we have, whenever $(\mathfrak{A},\bar{V})\models\Sigma(V)$, and letting $S_a=\{y\in A\mid(a,y)\in R^{\mathfrak{A}}\}$, that $(\mathfrak{A}\restriction S_a)\restriction_{\tau\cup\{E\}}$ is in C_1 or C_2, and $\bar{V}=\{a\in A\mid(\mathfrak{A}\restriction S_a)\restriction_{\tau\cup\{E\}}\in C_1\}$. This shows V is implicitely defined by $\Sigma(V)$. To show that V is not explicitely definable from $\Sigma(V)$ in $L_{\omega\omega}(\mathbf{Th})$, respectively in $L_{\infty\omega}(\mathbf{Th})$, it is enough to prove that the projection $C=\{\mathfrak{A}\mid\exists\bar{V}(\mathfrak{A},\bar{V})\models\Sigma(V)\}$ is not elementary in these logics. For this purpose we will utilize the structures $\mathfrak{A}_1(\kappa)\in C_1$, $\mathfrak{A}_2(\kappa)\notin C_1$, introduced in the proof of Theorem 1 to construct tree structures $D_1(\kappa)\notin C$, $D_2(\kappa)\in C$, such that for large enough κ

$$D_1(\kappa)\equiv D_2(\kappa)\quad\text{in}\quad L^m_{\infty\omega}(\mathbf{T}_\kappa),\tag{5}$$

respectively

$$D_1(\kappa)\equiv D_2(\kappa)\quad\text{in}\quad L_{\infty\omega}(\mathbf{T}_\kappa).\tag{6}$$

Let $(T,\bar{R})$ be the tree of all finite sequences of ordinals less than κ, where $\bar{R}$ is the immediate successor relation: $(x,y)\in\bar{R}\Leftrightarrow y=x-\langle\beta\rangle$ for some $\beta\in\kappa$, with the empty sequence $\emptyset$ as the root. We expand $(T,\bar{R})$ in two different ways to structures of type $\{R\}\cup\tau\cup\{E\}$. The sets $S_a=\{x\in T\mid(a,x)\in\bar{R}\}$, $a\in T$, have power κ and are mutually disjoint.

Expansion 1. Let $D_1(\kappa)$ be the result of giving to each S_a the structure of $\mathfrak{A}_1(\kappa)$, this is, taking $D_a=(D_1(\kappa)\restriction S_a)\restriction_{\tau\cup\{E\}}\approx\mathfrak{A}_1(\kappa)$ and interpreting the symbols of $\tau\cup\{E\}$ in $D_1(\kappa)$ as the union of their interpretations in the D_a. Obviously $D_1(\kappa)\notin C$ because $(D_1(\kappa),\bar{V})\models\Sigma(V)$ would imply $E^{D_1(\kappa)}\restriction S_\emptyset$ to be an equivalence relation in $S_\emptyset$ with set of representatives $\bar{V}\cap S_\emptyset$, and for $y\in S_\emptyset$, D_y would be in C_1 only if $y\in\bar{V}$, which is not the case by construction since the equivalence classes are infinite and all $D_y\in C_1$.

Expansion 2. Fix choice sets V_1 and V_2 for $E^{\mathfrak{A}_1(\kappa)}$ and $E^{\mathfrak{A}_2(\kappa)}$, respectively. We define inductively in the level of $a\in T$ a structure (D_a,V_a) of type $\tau\cup\{E,V\}$ with domain S_a.

First, $(D_\emptyset, V_\emptyset)\approx(\mathfrak{A}_1(\kappa), V_1)$. Assume (D_b, V_b) has been defined on S_b for each b of level $\leqq n$, then for a of level $n+1$ let b be its immediate predecessor in the tree and define: $(D_a, V_a)\approx(\mathfrak{A}_1(\kappa), V_1)$ if $a\in V_b$, $(D_a, V_a)\approx(\mathfrak{A}_2(\kappa), V_2)$ if $a\notin V_b$. $(D_2(\kappa), \bar{V})$ will be the expansion of $(T, \bar{R})$ which interprets $\tau\cup\{E, V\}$ by the union of its interpretation in the (D_a, V_a), taking care of adding the root $\emptyset$ to the interpretation of V. Then $(D_2(\kappa), \bar{V})\models\Sigma(V)$ by construction and so $D_2(\kappa)\in C$.

Equivalences (5) and (6), and so the proof of the theorem will follow from the next lemma.

Lemma 3. *If $\mathfrak{A}_1$ and $\mathfrak{A}_2$ have power at most $\kappa\geqq\omega$, and $\mathfrak{A}_1\equiv\mathfrak{A}_2$ in $L^\alpha_{\infty\omega}$, then $D_1(\kappa)\equiv D_2(\kappa)$ in $L^\alpha_{\infty\omega}(\mathbf{T}_\kappa)$.*

Proof. We already know that under the hypothesis: $\mathfrak{A}_1(\kappa)\equiv\mathfrak{A}_2(\kappa)$ in $L^\alpha_{\infty\omega}$. We define now a Karp system of partial isomorphisms $(I_\gamma)_{\gamma<\alpha}$ between $D_1(\kappa)$ and $D_2(\kappa)$ as follows: $f\in I_\gamma$ if and only if it is a partial map of finite domain from T_1 to T_2 such that:

i. Domain and range of f are closed under predecessors, and f is an isomorphism between the induced subtrees.

ii. If $\bar{a}=(a_1, \ldots, a_n)$ is a listing of dom f, and $\bar{b}=(b_1, \ldots, b_n)$ with $b_i=f(a_i)$, then for all $x\in\operatorname{dom} f$: $(D_x, \bar{a}\restriction S_x)\equiv(D_{f(x)}, \bar{b}\restriction S_{f(x)})$ in $L^{\gamma+1}_{\infty\omega}$.

For simplicity we will write the last equivalence as $\bar{a}\restriction S_x\overset{\gamma}{\equiv}\bar{b}\restriction S_{f(x)}$. Obviously, $\emptyset\in I_\gamma$ for any $\gamma<\alpha$. Now we show that this system has the extension property with respect to $\exists$. Assume $f\in I_{\gamma+1}$, $\gamma+1<\alpha$, and let $\bar{a}$ and $\bar{b}$ be as in (ii) above. Given $x\in T_1\backslash\operatorname{dom} f$ we must find $g\in I_\gamma$ such that $g\supseteqq f$ and $x\in\operatorname{dom} g$. Let $z_0, \ldots, z_t=x$ be the unique path in the tree connecting the root to x, and let z_k be the first in the path not belonging to dom f, then $z_{k+1}, \ldots, z_t\notin\operatorname{dom} f$. Let $a_u\in\operatorname{dom} f$ be the inmediate predecessor of z_k, since by definition we must have $\bar{a}\restriction S_{a_u}\overset{\gamma+1}{\equiv}\bar{b}\restriction S_{b_u}$, we may find $z'_k\in S_{b_u}$ such that $\bar{a}z_k\restriction S_{a_u}\overset{\gamma}{\equiv}\bar{b}z'_k\restriction S_{b_u}$ using and adequate formula of rank $\gamma+1$. Now find $z'_{k+1}\in S_{z'_k}$ such that $z_{k+1}\overset{\gamma}{\equiv}z'_{k+1}$; this is possible because by our initial observation $D_{z_k}\equiv D_{z'_k}$ in $L^{\gamma+2}_{\infty\omega}$. Inductively, continue choosing $z'_{i+1}\in S_{z'_i}$ such that $z_{i+1}\overset{\gamma}{\equiv}z'_{i+1}$ $i=k+1, \ldots, t$. Then $g=f\cup\{(z_i, z'_i)\mid i=k, \ldots, t\}\in I_\gamma$ because (i) holds by construction, and (ii) may be verified case by case: for $a_w\in\operatorname{dom}, f$, $w\neq u$,

$$\bar{a}\bar{z}\restriction S_{a_w}=\bar{a}\restriction S_{a_w}\overset{\gamma}{\equiv}\bar{b}\restriction S_{b_w}=\bar{b}\bar{z}'\restriction S_{b_w}$$

by hypothesis;

$$\bar{a}\bar{z}\restriction S_{a_u}=\bar{a}z_k\restriction S_{a_u}\overset{\gamma}{\equiv}\bar{b}z'_k\restriction S_{b_u}=\bar{b}\bar{z}'\restriction S_{b_u}$$

by choosing of z'_k. Now, for $i=k, \neq, t-1$,

$$\bar{a}\bar{z}\restriction S_{z_i}=z_{i+1}\overset{\gamma}{\equiv}z'_{i+1}=\bar{b}\bar{z}'\restriction S_{z'_i}$$

by construction, and

$$\bar{a}\bar{z}\restriction S_{z_t}=\emptyset\overset{\gamma}{\equiv}\emptyset=\bar{b}\bar{z}'\restriction S_{z'_t}$$

The other direction of the extension property is analogous.

With this we have shown that $D_1(\kappa) \equiv D_2(\kappa)$ in $L^{\alpha}_{\infty\omega}$, it remains to show that the κ-thin quantifiers are eliminable in $D_i(\kappa)$, $i=1,2$. Consider the Karp system of partial isomorphisms defined as before but from $D_1(\kappa)$ into itself, similarly for $D_2(\kappa)$.

Claim. Let E_1 be the interpretation of E in $D_1(\kappa)$, then for any $\bar{a} \in T^n$ closed under predecessors and $x_1, \ldots, x_r, y_1, \ldots, y_r \in E_1(x_1) \setminus \bar{a}$ with $x_i \neq x_j$, $y_i \neq y_j$, and any γ there is $f \in I_\gamma$ which is the identity in $\bar{a}$ and $f(x_i) = y_i$. A similar claim holds for $D_2(\kappa)$. To see this notice that under the hypothesis, the x_i and y_i lie in the same S_u. Close $\bar{a}u$ under predecessors to obtain $\bar{a}'$; still $x_i, y_i \notin \bar{a}'$. By Claim 1 in the proof of Theorem 1, applied to an appropriate complete formula of rank γ, we have: $(D_u, \bar{a}'\bar{x} \restriction S_u) \equiv (D_u, \bar{a}'\bar{y} \restriction S_u)$ in $L^{\gamma+1}_{\infty\omega}$; moreover, for $w \neq u$, $\bar{a}'\bar{x} \restriction S_w = \bar{a}' \restriction S_w = \bar{a}'\bar{y} \restriction S_w$. This shows condition (ii) for the map f which is the identity in $\bar{a}'$ and sends x_i to y_i; condition (i) holds by construction.

Under the same hypothesis of the Claim we have by the properties of Karp systems:

$$D_i(\kappa) \models \varphi[x_1, \ldots, x_r, \bar{a}] \leftrightarrow \varphi[y_1, \ldots, y_r, \bar{a}]$$

for any $\varphi \in L_{\infty\omega}$. Arguing as in the proof of Theorem 1 we may conclude that any formula of $L_{\infty\omega}(\mathbf{T}_\kappa)$ is equivalent in $D_i(\kappa)$ to a formula of $L_{\infty\omega}$ of the same quantifier rank which only depends on the power of $D_i(\kappa)$, and so it is the same for $D_1(\kappa)$ and $D_2(\kappa)$ which have both power κ. Hence, $D_1(\kappa) \equiv D_2(\kappa)$ in $L^{\alpha}_{\infty\omega}(\mathbf{T}_\kappa)$. □

Again, if Ch is definable in the logic we may avoid the relativization hypothesis to get a generalization of Friedman's theorem in [Fr].

Theorem 5. *No logic between $L_{\omega\omega}(Ch)$ or $L_{\omega\omega}(Q_\alpha)$ with $\alpha \geq 1$ and $L_{\infty\omega}(\mathbf{Th})$ satisfies Beth's definability theorem.*

Proof. Beth $L_{\omega\omega}(Q_\alpha) \nleq L_{\infty\omega}(\mathbf{Th})$ by Theorem 4 because it has relativizations and is not a Karp logic for $\alpha \geq 1$. Now the sentence $\Sigma(V)$ that defines V implicitely in $L_{\omega\omega}(Ch)$ but not explicitely in $L_{\infty\omega}(\mathbf{Th})$ is

$$\forall x[(Vx \to \theta \wedge \psi(x) \wedge \neg Chy(Rxy \wedge Vy)) \quad \wedge (\neg Vx \wedge \theta \to \psi(x) \wedge Chy(Rxy \wedge Vy))]$$

where θ says that E is an equivalence relation and $\psi(x)$ is the formula

$$\forall y(Rxy \to \exists! z(Rxz \wedge Vz \wedge Eyz)).$$

The proof is as in Theorem 4 starting with the structures $\mathfrak{A}_1$, $\mathfrak{A}_2$ introduced in the proof of Theorem 3. □

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